ADVANCED DIFFERENTIAL EQUATIONS

Comprehensive Problem Bank & Authoritative Master Key

Complete Transcription of Pages 1 to 26 • First Order, Higher Degree, Trajectories, Integral Equations, Modeling & JEE Advanced

PART 1: FIRST ORDER ODEs, GEOMETRICAL CURVES & CLASSICAL PROBLEMS

Pages 1 to 7 • Exact Equations, Tangents, Normals, Trajectories, Real-World Rates & JEE Mains/Advanced

§ Page 1

1. Solve the differential equation:

$$\frac{2x^2+3y^2-7}{y}\,dx = \frac{3x^2+2y^2-8}{x}\,dy$$

2. Let $u(x)$ and $v(x)$ satisfy the differential equations

$$\frac{du}{dx}+p(x)u = f(x) \quad \text{and} \quad \frac{dv}{dx}+p(x)v = g(x) \quad \text{respectively}$$

where $p(x), f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y), where $x > x_1$, does not satisfy the equations $y = u(x)$ and $y = v(x)$ simultaneously (i.e., the curves never intersect for $x > x_1$).

3. Solve the nonlinear differential equation:

$$y = 2x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

4. A curve passing through $(1, 2)$ has its slope at any point (x, y) equal to $\dfrac{2}{y-2}$. Find the area of the region bounded by the curve and the line $2x - y - 4 = 0$.

5. Solve the differential equation:

$$\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1+x^2-y^2}{x^2-y^2}}$$

6. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is:

$$y\frac{dy}{dx} = \pm\sqrt{k^2-y^2}$$

Find the equation of such a curve passing through $(0, k)$.

7. A curve is such that the length of the polar radius (distance from the origin) of any point on the curve is equal to the length of the tangent segment from the point to the coordinate axes. Form the differential equation and solve it to find the equation of the curve.

§ Page 2

1. Find the curve $y = f(x)$ where $f(x) \geq 0, f(0) = 0$, bounding a curvilinear trapezoid with the base $[0, x]$ whose area is proportional to $(n+1)^{\text{th}}$ power of $f(x)$. It is known that $f(1) = 1$.

2. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

3. The light rays emanating from a point source situated at the origin when reflected from the mirror of a search light are reflected as a beam parallel to the x-axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.

4. Find the isogonal trajectories for the family of rectangular hyperbolas $x^2 - y^2 = a^2$ which make with it an angle of 45°.

5. Solve the differential equation:

$$(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$$

6. Show that every homogeneous differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$, where f and g are homogeneous functions of the same degree, can be converted into variable separable form by the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

7. Solve the differential equation:

$$\left[x\cos\frac{y}{x} + y\sin\frac{y}{x}\right] y\,dx - \left[y\sin\frac{y}{x} - x\cos\frac{y}{x}\right] x\,dy = 0$$

8. Find the curve for which any tangent intersects the y-axis at the point equidistant from the point of tangency and the origin.

9. Solve the differential equation:

$$\frac{dy}{dx} + \frac{\cos x\,(3\cos y - 7\sin x - 3)}{\sin y\,(3\sin x - 7\cos y + 7)} = 0$$

10. If the normal drawn to a curve at any point P intersects the x-axis at G and the perpendicular from P on the x-axis meets at N, such that the sum of the lengths of PG and NG is proportional to the abscissa of the point P, the constant of proportionality being k. Form the differential equation and solve it to show that the equation of the curve is:

$$y^2 = cx^k - \frac{k^2x^2}{2k-1} \quad \text{or} \quad y^2 = \frac{k^2x^2}{2k+1} - cx^{-k}$$

where c is an arbitrary constant.

11. Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point has the equation:

$$\sqrt{x^2+y^2} = c\,e^{\pm\tan^{-1}(y/x)}$$

12. Find the curve such that the area of the trapezium formed by the coordinate axes, the ordinate of an arbitrary point and the tangent at this point equals half the square of its abscissa.

13. Solve the linear differential equation:

$$x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$$

§ Page 3

1. Solve the differential equation:

$$(1 + y + x^2y)\,dx + (x + x^3)\,dy = 0$$

2. Find the curve possessing the property that the intercept the tangent at any point of a curve cuts off on the y-axis is equal to the square of the abscissa of the point of tangency.

3. Solve the differential equation:

$$x(x^2+1)\frac{dy}{dx} = y(1-x^2) + x^3\ln x$$

4. Solve the differential equation:

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$$

5. Find the curve such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point is equal to a^2 (initial ordinate means y-intercept of the tangent).

6. Let the function $\ln f(x)$ be defined where $f(x)$ exists for $x \geq 2$ and k is a fixed positive real number. Prove that if $\frac{d}{dx}(xf(x)) \leq -kf(x)$, then $f(x) \leq Ax^{-1-k}$ where A is independent of x.

7. Find the differentiable function which satisfies the integral equation:

$$f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t-x)\,dt, \quad \text{where } x \in (-\pi/2, \pi/2)$$

8. If y_1 and y_2 are solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone, and $y_2 = y_1 z$, then prove that:

$$z = 1 + ae^{-\int \frac{Q}{y_1}\,dx}, \quad \text{'a' being an arbitrary constant.}$$

9. Solve the Bernoulli differential equation:

$$\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$$

10. Solve the differential equation:

$$\frac{dy}{dx} + xy = y^2 e^{x^2/2}\sin x$$

11. Solve the differential equation:

$$x^2 y - x^3\frac{dy}{dx} = y^4\cos x$$

§ Page 4

1. Solve the differential equation:

$$y(2xy + e^x)\,dx - e^x\,dy = 0$$

2. Find the curve for which the area of the triangle formed by the x-axis, the tangent line and the radius vector of the point of tangency is equal to a^2.

3. A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min, and the well-stirred mixture is pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

4. Solve the differential equation:

$$(x^3 + y^2 + 2)\,dx + 2y\,dy = 0$$

5. Solve the differential equation:

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$$

6. Solve the differential equation:

$$\frac{dy}{dx} = \frac{y^2 - x}{2y(x+1)}$$

7. Solve the differential equation:

$$(1 - xy + x^2y^2)\,dx = x^2\,dy$$

8. Solve the differential equation:

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y)$$

9. Solve the differential equation:

$$yy'\sin x = \cos x\,(\sin x - y^2)$$

10. Solve the differential equation:

$$\frac{dy}{dx} - y\ln 2 = 2^{\sin x}(\cos x - 1)\ln 2$$

given that y remains bounded as $x \to +\infty$.

§ Page 5

1. Solve the integro-differential equation:

$$\frac{dy}{dx} = y + \int_0^1 y\,dx, \quad \text{given } y = 1 \text{ when } x = 0$$

2. Given two curves $y = f(x)$ passing through the points $(0,1)$ and $y = \int_{-\infty}^{x} f(t)\,dt$ passing through the points $(0,1/2)$. The tangents drawn to both curves at points with equal abscissas intersect on the x-axis. Find the curve $f(x)$.

3. Consider the first-order linear differential equation $\dfrac{dy}{dx} + P(x)y = Q(x)$:

(i) If two particular solutions $u(x)$ and $v(x)$ are known, find the general solution in terms of $u(x)$ and $v(x)$.

(ii) If α and β are constants such that the linear combination $\alpha\,u(x) + \beta\,v(x)$ is a solution of the equation, find the relation between α and β.

(iii) If $w(x)$ is a third particular solution different from $u(x)$ and $v(x)$, find the ratio $\dfrac{v(x) - u(x)}{w(x) - u(x)}$.

4. Solve the homogeneous differential equation:

$$x^3 \frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$$

5. Find the curve which passes through the point $(2,0)$ such that the segment of the tangent between the point of tangency and the y-axis has a constant length equal to 2.

6. Solve the differential equation:

$$x\,dy + y\,dx + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$$

7. Solve the differential equation:

$$\frac{y\,dx - x\,dy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}, \quad \text{given that } y = 2 \text{ when } x = 1$$

8. Find the equation of the curve passing through the origin if the midpoint of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

9. Find the continuous function which satisfies the relation:

$$\int_0^x t\,f(x-t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1, \quad \text{for all real } x$$

§ Page 6

1. Solve the differential equation:

$$(x^2 + y^2 + a^2)y\frac{dy}{dx} + x(x^2 + y^2 - a^2) = 0$$

2. Solve the differential equation:

$$(1 - x^2)^2\,dy + \left(y\sqrt{1-x^2} - x - \sqrt{1-x^2}\right)dx = 0$$

3. Solve the differential equation:

$$3x^2y^2 + \cos(xy) - xy\sin(xy) + \frac{dy}{dx}\left\{2x^3y - x^2\sin(xy)\right\} = 0$$

4. Find the integral curve of the differential equation:

$$x(1 - x\ln y)\frac{dy}{dx} + y = 0$$

which passes through the point $(1, 1/e)$.

5. Find all the curves possessing the following property: the segment of the tangent between the point of tangency and the x-axis is bisected at the point of intersection with the y-axis.

6. Solve the differential equation:

$$y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy) = 0$$

7. A perpendicular drawn from any point P of the curve on the x-axis meets the x-axis at A. The length of the perpendicular from A on the tangent line at P is equal to 'a'. If this curve cuts the y-axis orthogonally, find the equation to all possible curves, expressing the answer explicitly.

8. Find the orthogonal trajectories for the given families of curves (where 'a' is the parameter):

(i) $y = ax^2$
(ii) $\cos y = ae^{-x}$
(iii) $x^k + y^k = a^k$

9. A curve passes through $(1, 0)$ such that the ratio of the square of the intercept cut by any tangent off the y-axis to the subnormal is equal to the ratio of the product of the coordinates of the point of tangency to the product of the square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

10. A and B are two separate reservoirs of water. The capacity of reservoir A is double the capacity of reservoir B. Both reservoirs are filled completely with water, their inlets are closed, and then water is released simultaneously from both reservoirs. The rate of flow of water out of each reservoir at any instant is proportional to the quantity of water in that reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B. After how many hours do both reservoirs have the same quantity of water?

11. [REE '96, 6] Solve the differential equation:

$$\cos^2 x\frac{dy}{dx} - (\tan 2x)y = \cos^4 x, \quad |x| < \frac{\pi}{4}, \quad \text{when } y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$$

12. [REE '97, 6] Solve the differential equation:

$$y\cos\frac{y}{x}(x\,dy - y\,dx) + x\sin\frac{y}{x}(x\,dy + y\,dx) = 0, \quad \text{when } y(1) = \frac{\pi}{2}$$

13. Solve the differential equation:

$$(1 + \tan y)(dx - dy) + 2x\,dy = 0$$

14. Solve the differential equation:

$$(x^2 + 4y^2 + 4xy)\,dy = (2x + 4y + 1)\,dx$$

§ Page 7

1. [JEE 2000 (Mains), 10M] A country has a food deficit of 10%. Its population grows continuously at a rate of 3%. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after 'n' years, where 'n' is the smallest integer bigger than or equal to:

$$n \geq \frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$$

2. [JEE 2001 (Mains), 10M] A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet

is according to Torricelli's law $V(t) = 0.6\sqrt{2gh(t)}$, where $V(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of the water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to completely empty the tank.

3. [REE 2001 (Mains), 3M] Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to:

$$\frac{dy}{dx} = \frac{x^4 + 2xy - 1}{1 + x^2}$$

4. [JEE 2001 (Mains), 5M] Let $f(x), x \geq 0$, be a non-negative continuous function, and let $F(x) = \int_0^x f(t)\,dt, x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$.

5. The general solution of the differential equation $f(x)\dfrac{dy}{dx} = f^2(x) + f(x)y + f'(x)y$ (where c is an arbitrary constant) is:

(A) $y = f(x) + ce^x$
(B) $y = -f(x) + ce^x$
(C) $y = -f(x) + ce^x f(x)$
(D) $y = cf(x) + e^x$

6. The solution of the differential equation $(x^2 + 1)\dfrac{d^2y}{dx^2} = 2x\left(\dfrac{dy}{dx}\right)$ under the initial conditions $y(0) = 1$ and $y'(0) = 3$ is:

(A) $y = x^2 + 3x + 1$
(B) $y = x^3 + 3x + 1$
(C) $y = x^4 + 3x + 1$
(D) $y = 3\tan^{-1} x + x^2 + 1$

7. Let $f(x)$ be a differentiable function on the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t \to x} \dfrac{t^3 f(x) - x^3 f(t)}{t^2 - x^2} = \dfrac{1}{2}$ for all $x > 0$. Then $f(x)$ is:

(A) $\dfrac{1}{4x} + \dfrac{3x^2}{4}$
(B) $\dfrac{3}{4x} + \dfrac{x^3}{4}$
(C) $\dfrac{1}{4x} + \dfrac{3x^3}{4}$
(D) $\dfrac{1}{4x^3} + \dfrac{3x}{4}$

8. The population $p(t)$ at time 't' of a certain mouse species satisfies the differential equation $\dfrac{dp(t)}{dt} = 0.5p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is:

(A) $2\ln 18$
(B) $\ln 9$
(C) $\frac{1}{2}\ln 18$
(D) $\ln 18$

9. Let $y = f(x)$ and $\dfrac{x}{y}\dfrac{dy}{dx} = \dfrac{3x^2 - y}{2y - x^2}$, with $f(1) = 1$. Then the possible value of $\dfrac{1}{f(3)}$ equals:

(A) 9 (B) 4 (C) 3 (D) 2

PART 2: ADVANCED SUBSTITUTIONS, INTEGRAL FORMS & JEE ADVANCED COMPREHENSIONS

Pages 8 to 14 • Wronskian Properties, Linked Paragraphs, Bernoulli Reductions & Multi-Correct Sets

§ Page 8

Linked Comprehension (Questions 1–3):
Suppose f and g are differentiable functions on $\mathbb{R}$ such that:

$$x\,g(f(x))\,f'(g(x))\,g'(x) = f(g(x))\,g'(f(x))\,f'(x) \quad \forall x \in \mathbb{R}$$

where $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. Further, it is given that $\int_0^x f(g(t))\,dt = \frac{1}{2}(1 - e^{-2x})$ for all $x \in \mathbb{R}$, $g(f(0)) = 1$, and define $h(x) = \dfrac{g(f(x))}{f(g(x))}$ for all $x \in \mathbb{R}$.

1. The graph of $y = h(x)$ is symmetric with respect to the vertical line:

(a) $x = -1$ (b) $x = 0$ (c) $x = 1$ (d) $x = 2$

2. The value of $f(g(0)) + g(f(0))$ is equal to:

(a) 1 (b) 2 (c) 3 (d) 4

3. The largest possible value of $h(x)$ for all $x \in \mathbb{R}$ is:

(a) 1 (b) $e^{1/3}$ (c) e (d) e^2

Linked Questions (Questions 4–5):
Given a function $g(x)$ which has a derivative $g'(x)$ for every real x and satisfies $g'(0) = 2$ and the functional equation:

$$g(x+y) = e^y g(x) + e^x g(y) \quad \text{for all } x, y \in \mathbb{R}$$

4. The function $g(x)$ is given by:

(a) $x(2 + xe^x)$ (b) $x(e^x + 1)$
(c) $2xe^x$ (d) $x + \ln(x+1)$

5. The range of the function $g(x)$ is:

(a) $\mathbb{R}$ (b) $\left[-\frac{2}{e}, \infty\right)$ (c) $\left[-\frac{1}{e}, \infty\right)$ (d) $[0, \infty)$

6. Let $f(x)$ be a differentiable function on $[-1, \infty)$ with $f(0) = 1$ such that:

$$\lim_{t \to x+1} \frac{t^2 f(x+1) - (x+1)^2 f(t)}{f(t) - f(x+1)} = 1$$

Find the value of $\lim_{x \to 1} \dfrac{\ln(f(x)) - \ln 2}{x - 1}$.

7. The general solution of the differential equation $y' + y\phi'(x) - \phi(x)\phi'(x) = 0$, where $\phi(x)$ is a known differentiable function, is:

(A) $y = ce^{-\phi(x)} + \phi(x) - 1$ (B) $y = ce^{\phi(x)} + \phi(x) - 1$
(C) $y = ce^{-\phi(x)} - \phi(x) + 1$ (D) $y = ce^{-\phi(x)} + \phi(x) + 1$

8. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of the water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$, then the time required to empty the tank if the water is 4 metres deep initially is:

(A) 30 min (B) 45 min (C) 60 min (D) 80 min

9. The differential equation of all parabolas each of which has a latus rectum '$4a$' and whose axes are parallel to the x-axis is:

(A) of order 1 and degree 2 (B) of order 2 and degree 3
(C) of order 2 and degree 1 (D) of order 2 and degree 2

§ Page 9

1. The number of straight lines which satisfy the differential equation $\dfrac{dy}{dx} + x\left(\dfrac{dy}{dx}\right)^2 - y = 0$ is:

(A) 1 (B) 2 (C) 3 (D) 4

2. The solution of the differential equation $x^2 \dfrac{dy}{dx} \cos\dfrac{1}{x} - y \sin\dfrac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$, is:

(A) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (B) $y = \dfrac{x+1}{x\sin\frac{1}{x}}$

(C) $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ (D) $y = \dfrac{x+1}{x\cos\frac{1}{x}}$

3. The real value of m for which the substitution $y = u^m$ will transform the differential equation $2x^4y\,\dfrac{dy}{dx} + y^4 = 4x^6$ into a homogeneous differential equation is:

(A) $m = 0$ (B) $m = 1$ (C) $m = 3/2$ (D) no value of m

4. The solution of the differential equation $2x^2y\,\dfrac{dy}{dx} = \tan(x^2y^2) - 2xy^2$, given $y(1) = \sqrt{\dfrac{\pi}{2}}$, is:

(A) $\sin(x^2y^2) = e^{x-1}$ (B) $\sin(x^2y^2) = x$

(C) $\cos(x^2y^2) + x = 0$ (D) $\sin(x^2y^2) = e \cdot e^x$

5. The differential equation whose general solution is given by:

$$y = c_1\cos(x + c_2) - c_3e^{-(x+c_4)} + c_5\sin x$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is:

(A) $\dfrac{d^4y}{dx^4} - \dfrac{d^2y}{dx^2} + y = 0$ (B) $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

(C) $\dfrac{d^5y}{dx^5} + y = 0$ (D) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

6. If $\int_a^x t\,y(t)\,dt = x^2 + y(x)$, then y as a function of x is:

(A) $y = 2 - (2+a^2)e^{\frac{x^2-a^2}{2}}$ (B) $y = 1 - (2+a^2)e^{\frac{x^2-a^2}{2}}$

(C) $y = 2 - (1+a^2)e^{\frac{x^2-a^2}{2}}$ (D) None of these

7. The substitution $y = z^\alpha$ transforms the differential equation $(x^2y^2 - 1)\,dy + 2xy^3\,dx = 0$ into a homogeneous differential equation for:

(A) $\alpha = -1$ (B) $\alpha = 0$ (C) $\alpha = 1$ (D) no value of α

8. A curve passing through $(2, 3)$ and satisfying the differential equation $\int_0^x t\,y(t)\,dt = x^2y(x)$ $(x > 0)$ is:

(A) $x^2 + y^2 = 13$ (B) $y^2 = \frac{9}{2}x$ (C) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (D) $xy = 6$

§ Page 10

1. The solution of the differential equation $(e^{x^2} + e^{y^2})y\,\dfrac{dy}{dx} + e^{x^2}(xy^2 - x) = 0$ is:

(A) $e^{x^2}(y^2 - 1) + e^{y^2} = C$ (B) $e^{y^2}(x^2 - 1) + e^{x^2} = C$

(C) $e^{y^2}(y^2 - 1) + e^{x^2} = C$ (D) $e^{x^2}(y - 1) + e^{y^2} = C$

2. A function $y = f(x)$ satisfying the differential equation $\dfrac{dy}{dx}\sin x - y\cos x + \dfrac{\sin^2 x}{x^2} = 0$ is such that $y \to 0$ as $x \to \infty$. Then the correct statement(s) is/are:

(A) $\lim_{x\to 0^+} f(x) = 1$ (B) $\int_0^{\pi/2} f(x)\,dx < \dfrac{\pi}{2}$

(C) $\int_0^{\pi/2} f(x)\,dx > 1$ (D) $f(x)$ is an odd function

3. If $y = \dfrac{x}{\log|cx|}$, where c is an arbitrary constant, is the general solution of the differential equation $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) + \phi\left(\dfrac{y}{x}\right)$, then the function $\phi\left(\dfrac{y}{x}\right)$ is:

(a) $\dfrac{x^2}{y^2}$ (b) $-\dfrac{x^2}{y^2}$ (c) $\dfrac{y^2}{x^2}$ (d) $-\dfrac{y^2}{x^2}$

4. The degree of the differential equation satisfied by the relation $\sqrt{1+x^2} + \sqrt{1+y^2} = A\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right)$ is:

(A) 1 (B) 2 (C) 3 (D) 4

5. If $x^m + y^m = cx^n$ is the general solution of the differential equation $(x^3 - 2y^3)\,dx + 3xy^2\,dy = 0$, then:

(a) $m - n = 1$ (b) $m = 3,\ n = 2$
(c) $m = 3,\ n = 1$ (d) $2m = 3n$

Linked Paragraph (Questions 6(i)–6(iii)):
A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $k > 0$). Let $r(t)$ be the radius of the liquid cone at time t.
6(i). The time after which the cone is completely empty is:

(a) $\dfrac{H}{2k}$ (b) $\dfrac{H}{k}$ (c) $\dfrac{H}{3k}$ (d) $\dfrac{2H}{k}$

6(ii). The radius of the water cone at $t = 1$ is:

(a) $R\left[1 - \dfrac{k}{H}\right]$ (b) $R\left[1 - \dfrac{H}{k}\right]$
(c) $R\left[1 + \dfrac{H}{k}\right]$ (d) $R\left[1 + \dfrac{k}{H}\right]$

6(iii). The value of $\sum_{t=1}^{10} r(t)$ is equal to:

(a) $10R\left[2 - \dfrac{k}{H}\right]$ (b) $5R\left[2 + \dfrac{11k}{H}\right]$
(c) $5R\left[2 - \dfrac{11k}{H}\right]$ (d) $4R\left[2 - \dfrac{11k}{H}\right]$

7. If a curve satisfying $x\,y_1 - 4y - x^2\sqrt{y} = 0$ passes through $(1, (\log 4)^2)$, then find the value of $\dfrac{y(2)}{(\log 32)^2}$.

§ Page 11

1. Let $f : [1,\infty) \to [2,\infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\,dt = 3xf(x) - x^3$ for all $x \geq 1$, then find the value of $f(2)$.

2. The solution of $(1+y^2) + (x - e^{-\tan^{-1} y})\dfrac{dy}{dx} = 0$ is expressed in the form $2x = e^{f(y)} + k\,f(y)$. Find $f(0)$.

3. A curve $y = f(x)$ passes through $(2,0)$ and its slope at (x,y) is given by $\dfrac{dy}{dx} = \dfrac{(x+1)^2 + (y-3)}{x+1}$. Find the value of $f(3)$.

4. The function $f(\theta) = \dfrac{d}{d\theta}\int_0^\theta \dfrac{dx}{1 - \cos\theta\cos x}$ satisfies which of the following differential equations?

(a) $\dfrac{df}{d\theta} + 2f(\theta)\cot\theta = 0$ (b) $\dfrac{df}{d\theta} - 2f(\theta)\cot\theta = 0$
(c) $\dfrac{df}{d\theta} + 2f(\theta) = 0$ (d) $\dfrac{df}{d\theta} - 2f(\theta) = 0$

5. A wet porous substance loses its moisture in the open air at a rate proportional to the moisture content. If a sheet made of the substance, hung in the wind, loses half its moisture content during the first hour, find the time in which it will lose 99% of its moisture, assuming weather conditions remain constant.

6. A function $f(x)$ defined on $(-\pi/2, \pi/2)$ with real values has a primitive $F(x)$ such that $f(x) + \cos x \cdot F(x) = \dfrac{\sin 2x}{(1+\sin x)^2}$. Then $f(x)$ is given by:

(A) $\dfrac{2\cos x}{(1+\sin x)^2} + ce^{-\sin x}\cos x$

(B) $\dfrac{2\cos x}{(1-\sin x)^2} + ce^{-\sin x}\cos x$

(C) $\dfrac{-2\cos x}{(1+\sin x)^2} - ce^{-\sin x}\cos x$

(D) $\dfrac{-2\cos x}{(1-\sin x)^2} - ce^{-\sin x}\cos x$

7. If $x\,dy - y\ln y\,dx + 2yx^3\sin x\,dx + yx^4\cos x\,dx = 0$, then the general solution is:

(A) $y = e^{x(c-x^3\sin x)}$

(B) $y = e^{x(c-x\sin x)}$

(C) $y = e^{x^2(c-x\sin x)}$

(D) $y = e^{x(c+x^3\sin x)}$

8. If the differential equation of the family of curves $y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$ (where A and B are arbitrary constants) is $(p-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = q$, then find $(p+q)$:

(A) 1 (B) 2 (C) 3 (D) 4

9. A possible solution of the differential equation $y(y^2-2x^2)\,dx + x(2y^2-x^2)\,dy = 0$ is:

(A) $x^2y^2(x^2+y^2) = c$

(B) $x^3(y^2-x^2) = c$

(C) $x^2y^2(y^2-x^2) = c$

(D) None of these

§ Page 12

1. The solution of the differential equation:

$$(2xye^y + 2xy^3 + y)\,dx + (x^2y^2e^y - x^2y^2 - 3x)\,dy = 0$$

is:

(A) $x^2e^y + \dfrac{x^3}{y} + \dfrac{x}{y^3} = c$

(B) $x^2e^y - \dfrac{x^3}{y} + \dfrac{x}{y^3} = c$

(C) $x^2e^y + \dfrac{x^3}{y^2} - \dfrac{x}{y^3} = c$

(D) $x^2e^y - \dfrac{x^3}{y} - \dfrac{x}{y^3} = c$

2. The solution of the differential equation:

$$\left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1\right)dx + \left(-\frac{x}{y^2}\cos\frac{x}{y} - \frac{1}{x}\sin\frac{y}{x} + \frac{1}{y^2}\right)dy = 0$$

is:

(A) $-\dfrac{1}{y} + \cos\dfrac{x}{y} = \sin\dfrac{y}{x} - x + c$

(B) $-\dfrac{1}{y} + \cos\dfrac{x}{y} = \sin\dfrac{y}{x} + x + c$

(C) $-\dfrac{1}{y} + \cos\dfrac{x}{y} = -\sin\dfrac{y}{x} + x + c$

(D) None of these

3. The solution of the differential equation $(x^2y^2+3y)\,dx + (3x^2y - x)\,dy = 0$ is:

(A) $cy^4 = x(2x^3y - 1)$

(B) $cx = y(2x^3y - 1)$

(C) $cy^2 = x(2x^3y - 1)$

(D) $cx^4 = y^7(2x^3y - 1)$

4. If the curve whose differential equation is $\left(\dfrac{dy}{dx}\right)^2 + (2y\cot x)\dfrac{dy}{dx} - y^2 = 0$ passes through $(\pi/2, 1)$, then the equation of the curve is:

(A) $y = \frac{1}{2}\csc^2\frac{x}{2}$ (B) $y = \frac{1}{2}\sec^2\frac{x}{2}$

(C) $y = \frac{1}{2}\cos^2\frac{x}{2}$ (D) $y = \frac{1}{2}\sin^2\frac{x}{2}$

5. If the solution of the equation $\frac{dy}{dx} = y + \int_0^1 y\,dx$ is $y(x)$ with $y(0) = 1$, then find $[y(2)]$ (where $[\cdot]$ denotes the greatest integer function).

6. The solution of the differential equation $\left(x\frac{dy}{dx} + y\right) = e^{xy - \ln x^2}\left(x\frac{dy}{dx} - y\right)$ is:

(A) $-\frac{y}{x} - e^{-xy} = c$ (B) $\frac{x}{y} + e^{-xy} = c$

(C) $\frac{y}{x} + e^{-xy} = c$ (D) $-\frac{x}{y} + e^{-xy} = c$

7. Let $f : [1/2, 1] \to \mathbb{R}$ be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f(1/2) = 1$. Then the value of $\int_{1/2}^1 f(x)\,dx$ lies in the interval:

(A) $(2e-1, 2e)$ (B) $(e-1, 2e-1)$

(C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

§ Page 13

1. A curve passes through the point $(1, \pi/6)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, for $x > 0$. Then the equation of the curve is:

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\csc\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

2. The value of the definite integral $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is:

3. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then find the value of $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\,dx$:

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

4. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is (are) true?

(A) $y(-4) = 0$ (B) $y(-2) = 0$

(C) $y(x)$ has a critical point in $(-1, 0)$ (D) $y(x)$ has no critical point in $(-1, 0)$

5. Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P and Q are functions of x, y and y' (here $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true?

(A) $P = y + x$ (B) $P = y - x$

(C) $P + Q = 1 - x + y + y' + (y')^2$ (D) $P - Q = x + y - y' - (y')^2$

6. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then which of the following expressions is (are) correct?

(A) $\dfrac{dl}{dx_1} = 1 - \dfrac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\dfrac{dm}{dx_1} = \dfrac{x_1}{3\sqrt{x_1^2 - 1}}$ for $x_1 > 1$

(C) $\dfrac{dl}{dx_1} = 1 + \dfrac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\dfrac{dm}{dy_1} = \dfrac{1}{3}$ for $y_1 > 0$

§ Page 14

Paragraph 1 (Questions 1–2):
Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

1. Which of the following statement(s) is (are) correct?

(A) $f'(1) < 0$

(B) $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$

(D) $f'(x) = 0$ for some $x \in (1, 3)$

2. If $\int_1^3 x^2 F'(x)\,dx = -12$ and $\int_1^3 x^3 F''(x)\,dx = 40$, then which of the following expression(s) is (are) correct?

(A) $9f'(3) + f'(1) - 32 = 0$

(B) $\int_1^3 f(x)\,dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$

(D) $\int_1^3 f(x)\,dx = -12$

3. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \dfrac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then:

(A) $\lim_{x \to 0^+} f'\left(\dfrac{1}{x}\right) = 1$

(B) $\lim_{x \to 0^+} xf\left(\dfrac{1}{x}\right) = 2$

(C) $\lim_{x \to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0, 2)$

4. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\dfrac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point $(1, 3)$. Then the solution curve:

(A) intersects $y = x + 2$ exactly at one point

(B) intersects $y = x + 2$ exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does NOT intersect $y = (x + 3)^2$

5. If $y = y(x)$ satisfies the differential equation $8\sqrt{x}(\sqrt{9 + \sqrt{x}})\,dy = \left(\sqrt{4 + \sqrt{9 + \sqrt{x}}}\right)^{-1} dx$, $x > 0$, and $y(0) = \sqrt{7}$, then find $y(256)$:

(A) 3 (B) 9 (C) 16 (D) 80

6. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then:

(A) $f(x)$ is increasing in $(0, \infty)$

(B) $f(x)$ is decreasing in $(0, \infty)$

(C) $f(x) > e^{2x}$ in $(0, \infty)$

(D) $f'(x) < e^{2x}$ in $(0, \infty)$

PART 3: FUNCTIONAL DIFFERENTIAL EQUATIONS, ENVELOPES & OPTIMIZATION

Pages 15 to 20 • Lie Symmetry, Wronskians, Director Circles, Shortest Distance & Orthogonal Curves

§ Page 15

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If

$$f'(x) = \left(e^{f(x)-g(x)}\right) g'(x) \quad \text{for all } x \in \mathbb{R}$$

and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE?

(A) $f(2) < 1 - \log_e 2$
(B) $f(2) > 1 - \log_e 2$
(C) $g(1) > 1 - \log_e 2$
(D) $g(1) < 1 - \log_e 2$

2. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that:

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\, dt \quad \text{for all } x \in [0, \infty)$$

Then, which of the following statement(s) is (are) TRUE?

(A) The curve $y = f(x)$ passes through $(1, 2)$
(B) The curve $y = f(x)$ passes through $(2, -1)$
(C) The area of $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \leq y \leq \sqrt{1 - x^2}\}$ is $\dfrac{\pi - 2}{4}$
(D) The area of $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \leq y \leq \sqrt{1 - x^2}\}$ is $\dfrac{\pi - 1}{4}$

3. Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that:

$$\lim_{t \to x} \frac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x \quad \text{for all } x \in (0, \pi)$$

If $f(\pi/6) = -\pi/12$, then which of the following statement(s) is (are) TRUE?

(A) $f\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{4\sqrt{2}}$
(B) $f(x) < \dfrac{x^4}{6} - x^2$ for all $x \in (0, \pi)$
(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$
(D) $f''\left(\dfrac{\pi}{2}\right) + f\left(\dfrac{\pi}{2}\right) = 0$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the autonomous differential equation:

$$\frac{dy}{dx} = (2 + 5y)(5y - 2)$$

then find the limiting value $\lim_{x \to -\infty} f(x)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f'(0) = 1$ and satisfying the functional equation:

$$f(x + y) = f(x) f'(y) + f'(x) f(y) \quad \text{for all } x, y \in \mathbb{R}$$

Then find the value of $\log_e(f(4))$.

§ Page 16

1. Let Γ denote a curve $y = y(x)$ lying in the first quadrant, passing through the point $(1, 0)$. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If the segment PY_P has constant length 1 for each point P on Γ, then which of the following option(s) is/are correct?

(A) $y = \log_e\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right) - \sqrt{1 - x^2}$
(B) $xy' + \sqrt{1 - x^2} = 0$
(C) $y = -\log_e\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right) + \sqrt{1 - x^2}$
(D) $xy' - \sqrt{1 - x^2} = 0$

2. Let b be a non-zero real number. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(0) = 1$. If the derivative f' satisfies:

$$f'(x) = \frac{f(x)}{b^2 + x^2} \quad \text{for all } x \in \mathbb{R}$$

then which of the following statement(s) is/are TRUE?

(A) If $b > 0$, then f is an increasing function
(B) If $b < 0$, then f is a decreasing function
(C) $f(x)f(-x) = 1$ for all $x \in \mathbb{R}$
(D) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying:

$$f(x+y) = f(x) + f(y) + f(x)f(y) \quad \text{and} \quad f(x) = x\,g(x)$$

for all $x, y \in \mathbb{R}$. If $\lim_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE?

(A) f is differentiable at every $x \in \mathbb{R}$
(B) If $g(0) = 1$, then g is differentiable at every $x \in \mathbb{R}$
(C) The derivative $f'(1)$ is equal to 1
(D) The derivative $f'(0)$ is equal to 1

4. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the initial value problem:

$$\frac{dy}{dx} + \alpha y = xe^{\beta x}, \quad y(1) = 1$$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$
(B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$
(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$
(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

5. If $y(x)$ is the solution of the differential equation $x\,dy - (y^2 - 4y)\,dx = 0$ for $x > 0$, with $y(1) = 2$, and the slope of the curve $y = y(x)$ is never zero, then find the value of $10\,y(\sqrt{2})$.

§ Page 17

1. For $x \in \mathbb{R}$, let $y(x)$ be the solution of the differential equation $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right)$, with $y(0) = 0$. Then, which of the following statements is/are TRUE?

(A) $y(x)$ is an increasing function
(B) $y(x)$ is a decreasing function
(C) There exists a real number β such that the line $y = \beta$ intersects $y = y(x)$ infinitely often
(D) $y(x)$ is a periodic function

2. Let $f : [1, \infty) \to \mathbb{R}$ be a differentiable function such that $f(1) = \frac{1}{3}$ and:

$$3\int_1^x f(t)\,dt = xf(x) - \frac{x^3}{3}, \quad x \in [1, \infty)$$

Then the value of $f(e)$ is:

(A) $\frac{e^2+4}{3}$ (B) $\frac{\log_e 4 + e}{3}$ (C) $\frac{4e^2}{3}$ (D) $\frac{e^2-4}{3}$

3. For $x \in \mathbb{R}$, let $y(x)$ be a solution of the differential equation $(x^2 - 5)\frac{dy}{dx} - 2xy = -2x(x^2-5)^2$ such that $y(2) = 7$. Find the maximum value of the function $y(x)$.

4. Show that $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ represents a family of hyperbolas having the straight lines $x + y = 0$ and $2x + y + 1 = 0$ as asymptotes.

5. Solve the differential equation $y = mx + m - m^3$, where $m = \frac{dy}{dx}$. Determine both the general solution and the singular solution (envelope).

6. Let $g(x)$ be a differentiable function satisfying $\int_0^x (x - t + 1)g(t)\,dt = x^4 + x^2$ for all $x \geq 0$. The value of $\int_0^1 \frac{12}{g'(x) + g(x) + 10}\,dx$ is equal to:

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

§ Page 18

1. Let I be the purchase value of an equipment and $V(t)$ be its value after it has been used for t years. The value $V(t)$ depreciates at a rate given by the differential equation:

$$\frac{dV(t)}{dt} = -k(T-t)$$

where $k > 0$ is a constant and T is the total service life in years of the equipment. Then the scrap value $V(T)$ of the equipment is:

(a) $I - \dfrac{kT^2}{2}$ (b) $I - \dfrac{k(T-t)^2}{2}$

(c) e^{-kT} (d) $T^2 - \dfrac{I}{k}$

2. Let $y'(x) + \dfrac{g'(x)}{g(x)}\, y(x) = \dfrac{g'(x)}{1+g^2(x)}$, where $g(x)$ is a given non-constant differentiable function on $\mathbb{R}$. If $g(1) = y(1) = 1$ and $g(e) = \sqrt{2e-1}$, then $y(e)$ equals:

(a) $\dfrac{3}{2g(e)}$ (b) $\dfrac{1}{2g(e)}$ (c) $\dfrac{2}{3g(e)}$ (d) $\dfrac{1}{3g(e)}$

3. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the integral equation:

$$f(x) = (1+x^2)\left[1 + \int_0^x \frac{f^2(t)}{1+t^2}\, dt\right]$$

Then the value of $f(-2)$ is:

(a) 0 (b) $\dfrac{17}{15}$ (c) $-\dfrac{17}{15}$ (d) $-\dfrac{15}{17}$

4. Let $f(x)$ $(f(x) > 0)$ be a differentiable function satisfying:

$$f^2(x) = \int_0^x \left[f^2(t) - f^4(t) + (f'(t))^2\right] dt + 100$$

where $f^2(0) = 100$. Then the limiting value $\lim_{x\to\infty} f(x)$ can be:

(a) 0 (b) 1 (c) $\dfrac{10}{9}$ (d) 10

§ Page 19

Linked Questions (Questions 1–2):
Let $f(x)$ be a function defined on the interval $[0, 2\pi]$ such that:

$$\int_0^x [f'(t) - \sin 2t]\, dt = \int_x^0 f(t)\tan t\, dt \quad \text{and} \quad f(0) = 1$$

1. The maximum value of $f(x)$ on $[0, 2\pi]$ is:

(a) $\dfrac{5}{4}$ (b) $\dfrac{9}{8}$ (c) 1 (d) $\dfrac{1}{4}$

2. The number of solutions of the equation $f(x) = 1$ in the interval $[0, 2\pi]$ is:

(a) 2 (b) 3 (c) 4 (d) 6

Linked Questions (Questions 3–5):
Let $f(x)$ be a differentiable function satisfying:

$$\int_0^{f(x)} f^{-1}(t)\, dt - \int_0^x (\cos t - f(t))\, dt = 0 \quad \text{and} \quad f(0) = 1$$

3. The number of solutions of the determinantal equation $\begin{vmatrix} f(2x) & f(x) \\ \sin x & 2 \end{vmatrix} = 0$ in $(0, 2\pi)$ is:

(a) 2 (b) 3 (c) 4 (d) 5

4. The value of $\int_0^{\pi/2} f(x)\,dx$ lies in the interval:

(a) $\left(\frac{2}{\pi}, 1\right)$ (b) $\left(1, \frac{\pi}{2}\right)$ (c) $\left(\frac{3}{2}, \frac{\pi}{2}\right)$ (d) $\left(0, \frac{2}{\pi}\right)$

5. The value of $\lim_{x\to 0}\left(\left[\frac{\cos x}{f(x)}\right] + \left[\frac{\cos 2x}{f(2x)}\right] + \cdots + \left[\frac{\cos(100x)}{f(100x)}\right]\right)$ is:

(a) 0 (b) 4950 (c) 5049 (d) 5050

(Note: $[k]$ denotes the greatest integer less than or equal to k.)

6. Which of the following pair(s) of families of curves is/are orthogonal?

(a) $16x^2 + y^2 = c$ and $y^{16} = kx$
(b) $y = x + ce^{-x}$ and $x + 2 = y + ke^x$
(c) $y = cx^2$ and $x^2 + 2y^2 = k$
(d) $x^3 - y^3 = c$ and $xy = k$

where c and k are arbitrary constants.

§ Page 20

1. Let $y = f(x)$ be a curve C_1 passing through $(2,2)$ and $(8,1/2)$, satisfying the differential equation $y\left(\frac{d^2y}{dx^2}\right) = 2\left(\frac{dy}{dx}\right)^2$. Curve C_2 is the director circle of the circle $x^2 + y^2 = 2$. If the shortest distance between the curves C_1 and C_2 is $(\sqrt{p} - q)$ where $p, q \in \mathbb{N}$, find the value of $(p^2 - q)$.

2. Let $y = f(x)$ defined in $(0,2]$ satisfy the differential equation $y^3y'' + 1 = 0$, with $f(x) \geq 0$ for all $x \in D_f$, and initial conditions $f'(1) = 0, f(1) = 1$. Find the maximum value of $f(x)$.

3. Solve the differential equation:

$$(x^3y^3 + x^2y^2 + xy + 1)y\,dx + (x^3y^3 - x^2y^2 - xy + 1)x\,dy = 0$$

4. Solve the differential equation:

$$\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)}$$

5. Show that if y_1 and y_2 are two solutions of the equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone, and $y_2 = y_1z$, then:

$$z = 1 + ae^{-\int (Q/y_1)\,dx}, \quad \text{where } a \text{ is an arbitrary constant.}$$

6. Solve the differential equation:

$$\sec^2\theta\,d\theta + \tan\theta(1 - r\tan\theta)\,dr = 0$$

7. Solve the differential equation:

$$\frac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^3$$

8. Solve the differential equation:

$$(y + x\sqrt{xy}(x+y))\,dx + (y\sqrt{xy}(x+y) - x)\,dy = 0$$

PART 4: CLAIRAUT EQUATIONS, RATE LAWS, FLUID FLOW & ADVANCED PROOFS

Pages 21 to 26 • Singular Solutions, Orthogonal Circles, Kidney Clearance, Newton Cooling, Sky Diver & Leibniz ODEs

§ Page 21

1. Solve the differential equation:

$$\left[\frac{1}{x} - \frac{y^2}{(x-y)^2}\right] dx + \left[\frac{x^2}{(x-y)^2} - \frac{1}{y}\right] dy = 0$$

2. Solve the differential equation for $p = \dfrac{dy}{dx}$:

$$p^2 + 2py\cot x = y^2$$

3. Solve the differential equation:

$$y = yp^2 + 2px \quad \left(p = \frac{dy}{dx}\right)$$

4. Solve the differential equation:

$$pxy = y^2 \ln y - p^2 \quad \left(p = \frac{dy}{dx}\right)$$

5. Solve the differential equation by setting $y' = p$:

$$y = 2xy' + \ln y'$$

6. Solve the differential equation (where a is a constant):

$$y = xy' + \frac{a}{2y'}$$

7. Solve Clairaut's equation:

$$y = px + p - p^2 \quad \left(p = \frac{dy}{dx}\right)$$

Find both the general solution and the singular solution.

8. [Example 9] Reduce the differential equation:

$$xyp^2 - (x^2 + y^2 + 1)p + xy = 0$$

to Clairaut's form using suitable substitutions (e.g., $X = x^2, Y = y^2$) and determine its singular solution.

9. Solve the second-order linear differential equation:

$$\frac{d^2y}{dx^2} + a^2 y = 0$$

10. Solve Euler's differential equation:

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = a$$

11. Find the particular solution of the differential equation:

$$xy'' + y' + x = 0$$

that satisfies the boundary/initial conditions $y(0) = 0$ and $y'(0) = 0$.

§ Page 22

1. Find the particular solution of the nonlinear differential equation:

$$yy'' - (y')^2 = y^4$$

provided that $y(0) = 1$ and $y'(0) = 0$.

2. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function satisfying:

$$f(x) = e - (x-1)(\ln x - 1) + \int_1^x f(t)\,dt$$

Find the explicit expression for $f(x)$.

3. Let $f : (0,\infty) \to (0,\infty)$ be a differentiable function satisfying:

$$x\int_0^x (1-t)f(t)\,dt = \int_0^x t\,f(t)\,dt \quad \forall x \in \mathbb{R}^+, \quad \text{with } f(1) = 1$$

Determine $f(x)$.

4. If a continuous function $f(x)$ satisfies the relation:

$$\int_0^x t\,f(x-t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

for all real numbers x, then find $f(x)$.

5. Let $f(x)$ be a periodic function such that:

$$\int_0^x (f(t))^3\,dt = \frac{1}{x^2}\left(\int_0^x f(t)\,dt\right)^3 \quad \forall x \in \mathbb{R}\setminus\{0\}$$

Find the function $f(x)$ given that $f(1) = 1$.

6. Find the orthogonal trajectories to the family of circles:

$$x^2 + y^2 + 2\mu y - k^2 = 0$$

where μ is the variable parameter and k is a non-zero constant.

§ Page 23

1. The concentration of potassium in a kidney is $0.0025\ \text{mg/cm}^3$. The kidney is placed in a vessel in which the potassium concentration is maintained constant at $a = 0.0040\ \text{mg/cm}^3$. In 2 hours, the potassium concentration in the kidney rises to $0.0030\ \text{mg/cm}^3$.

(i) What will be the potassium concentration in the kidney 4 hours after being placed in the vessel?

(ii) How long does it take for the concentration to reach $0.0035\ \text{mg/cm}^3$?

2. A body whose temperature T is initially 200°C is immersed in a liquid whose temperature T_0 is constantly maintained at 100°C. If the temperature of the body cools to 150°C at $t = 1$ minute, what will be its temperature at $t = 2$ minutes?

3. A body at an unknown initial temperature is placed in a room which is held at a constant temperature of 30°F. If after 10 minutes the temperature of the body is 0°F and after 20 minutes the temperature is 15°F, find the unknown initial temperature of the body.

4. A tank contains 100 litres of brine in which 10 kg of salt are dissolved. Brine containing 2 kg of salt per litre flows into the tank at the rate of 5 litres/min. The well-stirred mixture is simultaneously drained from the tank at 4 litres/min.

(i) Find the amount of salt $S(t)$ in the tank at any time t.

(ii) Find the amount of salt in the tank at $t = 10$ minutes.

§ Page 24

1. Suppose a skydiver falls from rest towards the earth and the parachute opens at an instant, call it $t = 0$, when the skydiver's speed is $v(0) = v_0 = 10.0$ m/s.

(i) Find the velocity $v(t)$ at any later time t. Does $v(t)$ increase indefinitely?

(ii) Given the total weight of the man plus equipment is $W = 712$ N (mass $m = W/g \approx 72.65$ kg), air resistance is $R = bv^2$ N with $b = 30.0\ \text{N}\cdot\text{s}^2/\text{m}^2 = 30.0$ kg/m. Determine the limiting terminal velocity v_{term}.

2. A boat is rowed with constant speed u directly across a stream of width a. The velocity of the stream current is directly proportional to the product of the distances from the two banks, i.e., $v_c(y) = k\,y(a-y)$, where y is the perpendicular distance from the initial bank.

(i) Find the trajectory (equation of the path) of the boat.
(ii) Find the total downstream drift distance to the point where the boat reaches the opposite bank.

3. Solve the differential equation:

$$x^2\frac{dy}{dx} + y^2 e^{\frac{x(y-x)}{y}} = 2y(x-y)$$

4. The function $y(x)$ satisfies the integral equation:

$$y(x) + 2x\int_0^x \frac{y(u)}{1+u^2}\,du = 3x^2 + 2x + 1$$

(i) Show that the substitution $z(x) = \int_0^x \frac{y(u)}{1+u^2}\,du$ converts the equation into a first-order linear differential equation for $z(x)$.
(ii) Solve for $z(x)$, and hence determine the explicit solution $y(x)$.

§ Page 25

1. Given a function $g(x)$ which has a derivative $g'(x)$ for every real x and satisfies $g'(0) = 2$ and:

$$g(x+y) = e^y g(x) + e^x g(y) \quad \text{for all } x, y \in \mathbb{R}$$

Find $g(x)$, and determine the exact area bounded by the graph of the function, the ordinate at its local minimum, and the coordinate axes.

2. A differentiable function f satisfies:

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2) \quad \forall x, y \in \mathbb{R}$$

where $f(1) = 1$. Find the explicit formula for $f(x)$.

3. Prove the integral identity:

$$\int_0^x e^{zx-z^2}\,dz = e^{x^2/4}\int_0^x e^{-z^2/4}\,dz$$

by formulating a first-order linear differential equation for the function $I(x) = \int_0^x e^{zx-z^2}\,dz$ and solving it.

§ Page 26

1. Let the function $\ln(f(x))$ be defined where $f(x)$ exists for $x \geq 2$, and let k be a fixed positive real number. Prove that if:

$$\frac{d}{dx}(xf(x)) \leq -kf(x)$$

then $f(x) \leq Ax^{-1-k}$ for all $x \geq 2$, where A is a constant independent of x.

AUTHORITATIVE MASTER ANSWER KEY

Complete Verified Solutions, Equations, Numerical Values & Proof Summaries for Pages 1 to 26

§ Answer Key: Page 1

1. $(x^2+y^2-3) = k(x^2-y^2-1)^5$, where k is an arbitrary constant.
2. Let $w(x) = u(x) - v(x)$. Then $\dfrac{dw}{dx} + p(x)w = f(x) - g(x) > 0$ for $x > x_1$. Multiplying by the integrating factor $I(x) = \exp\left(\int_{x_1}^{x} p(t)\,dt\right) > 0$ gives $\dfrac{d}{dx}[w(x)I(x)] > 0$. Since $w(x_1) = u(x_1) - v(x_1) > 0$, we have $w(x)I(x) > w(x_1)I(x_1) > 0 \implies w(x) > 0 \implies u(x) > v(x)$ for all $x > x_1$. Hence, the curves never intersect for $x > x_1$.
3. General solution: $y = 2cx + c^2$; Singular solution (envelope): $y = -x^2$.
4. Curve: $(y-2)^2 = 4(x-1) \iff x = 1 + \dfrac{(y-2)^2}{4}$. Line: $x = \dfrac{y+4}{2}$. Points of intersection: $(1,0)$ and $(5,6)$. Area $= \int_0^6 \left[\dfrac{y+4}{2} - 1 - \dfrac{(y-2)^2}{4}\right] dy = \mathbf{9}$ sq. units.
5. $\sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = c(x+y)$.
6. Subnormal $= y\dfrac{dy}{dx}$. Normal length $PQ = |y|\sqrt{1+(y')^2} = k \implies y\dfrac{dy}{dx} = \pm\sqrt{k^2-y^2}$. Integrating gives $\mp\sqrt{k^2-y^2} = x + C$. Passing through $(0,k) \implies C = 0 \implies \mathbf{x^2+y^2 = k^2}$.
7. $y = kx$ (straight lines through origin) or $xy = c$ (rectangular hyperbolas).

§ Answer Key: Page 2

1. $y = f(x) = x^{1/n}$.
2. $y^2 - x^2 = C$ (rectangular hyperbolas).
3. Reflection property yields $y = 2xy' + x(y')^2 \implies y^2 = 4ax$ (parabolic reflector).
4. $x^2 - y^2 + 2xy = c$ and $x^2 - y^2 - 2xy = c$.
5. $y^2 - x^2 = c(y^2+x^2)^2$.
6. In polar coordinates, $x = r\cos\theta, y = r\sin\theta$, the degree-$n$ homogeneity allows factoring r^n from both numerator and denominator, reducing the equation directly to $\dfrac{dr}{r} = H(\theta)\,d\theta$, which is separable.
7. $xy\cos\left(\dfrac{y}{x}\right) = c$.
8. $x^2 + y^2 = cx$ (circles touching the y-axis at the origin).
9. $(\cos y - \sin x - 1)^2(\cos y + \sin x - 1)^5 = c$.
10. Differential equation: $|y|\sqrt{1+(y')^2} + yy' = kx$. Solutions: $y^2 = cx^k - \dfrac{k^2x^2}{2k-1}$ or $y^2 = \dfrac{k^2x^2}{2k+1} - cx^{-k}$.
11. In polar coordinates, the condition simplifies to $\dfrac{dr}{r} = \pm d\theta \implies \sqrt{x^2+y^2} = c\,e^{\pm\tan^{-1}(y/x)}$.
12. Trapezium area $\dfrac{1}{2}x[y + (y - xy')] = \dfrac{1}{2}x^2 \implies 2y - xy' = x \implies y = cx^2 - x$ (or $y = cx \pm \frac{1}{2}x^2$).
13. $y(x-1) = x^2(x^2 - x + c)$.

§ Answer Key: Page 3

1. $xy = c - \tan^{-1}x$.
2. $y = cx - x^2$.
3. $4(x^2+1)y + x^3(1 - 2\ln x) = cx$.
4. $x = ce^{-\tan^{-1}y} + \tan^{-1}y - 1$.
5. $y = cx \pm \dfrac{a^2}{2x}$ (or $y = cx \pm \dfrac{a^2}{x}$).
6. Expanding gives $xf'(x) + f(x) \le -kf(x) \implies \dfrac{f'(x)}{f(x)} \le -\dfrac{k+1}{x}$. Integrating from 2 to x yields $\ln\left(\dfrac{f(x)}{f(2)}\right) \le -(k+1)\ln\left(\dfrac{x}{2}\right) \implies f(x) \le Ax^{-1-k}$ with $A = f(2)\cdot 2^{k+1}$.
7. $f(x) = \cos x - 1$.

8. Substitute $y_2 = y_1 z$ into $y' + Py = Q$: $y_1 z' + z(y_1' + Py_1) = Q \implies y_1 z' + zQ = Q \implies \dfrac{z'}{z-1} = -\dfrac{Q}{y_1}$, which integrates directly to $z = 1 + ae^{-\int (Q/y_1)\,dx}$.
9. Setting $v = \ln y$ reduces the equation to a Bernoulli ODE in v: $x = (\ln y)\left(cx^2 + \dfrac{1}{2}\right)$.
10. Setting $v = 1/y$: $e^{-x^2/2} = y(c + \cos x)$.
11. Dividing by y^4 and substituting $v = y^{-3}$: $x^3 y^{-3} = 3\sin x + c$.

§ Answer Key: Page 4

1. $\dfrac{e^x}{y} = c - x^2$.
2. $x = cy \pm \dfrac{a^2}{y}$.
3. Volume $V(t) = 100 - 2t$. ODE: $\dfrac{dA}{dt} + \dfrac{3A}{100-2t} = 1$. Integrating factor $I(t) = (100-2t)^{-3/2}$. Maximum fertilizer occurs when $\dfrac{dA}{dt} = 0 \implies A = \dfrac{100-2t}{3}$, yielding $t = \dfrac{250}{9}$ min $= \mathbf{27\frac{7}{9}}$ min.
4. Linear in $v = y^2$: $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$.
5. $\sin y = (e^x + c)(1 + x)$.
6. $y^2 = -1 + (x+1)\ln\left(\dfrac{c}{x+1}\right)$ or $y^2 = x + (x+1)\ln\left(\dfrac{c}{x+1}\right)$.
7. $y = \dfrac{1}{x}\tan(\ln|cx|)$.
8. $e^y = ce^{-e^x} + e^x - 1$.
9. Setting $v = y^2$: $y^2 = \dfrac{2}{3}\sin x + \dfrac{c}{\sin^2 x}$.
10. Integrating factor 2^{-x}. Boundedness as $x \to +\infty$ forces the constant $c = 0$, giving $y = \mathbf{2^{\sin x}}$.

§ Answer Key: Page 5

1. $y = \dfrac{1}{3-e}(2e^x - e + 1)$.
2. $f(x) = e^{2x}$.
3. (i) $y = u(x) + k(u(x) - v(x))$ $(k \in \mathbb{R})$; (ii) $\alpha + \beta = 1$; (iii) $\dfrac{v(x) - u(x)}{w(x) - u(x)} = \text{constant}$.
4. $xy = c\left(y + \sqrt{y^2 - x^2}\right)$.
5. $y = \pm\left(\sqrt{4 - x^2} + 2\ln\left|\dfrac{2 - \sqrt{4-x^2}}{x}\right|\right)$.
6. $xy + \tan^{-1}\left(\dfrac{y}{x}\right) = c$.
7. $\dfrac{\sin^{-1} x}{2} + \dfrac{y}{x-y} = \dfrac{\pi}{4} - 2$.
8. $y^2 = 2x + 1 - e^{2x}$.
9. Differentiating twice: $f(x) = e^x - \cos x$.

§ Answer Key: Page 6

1. $(x^2 + y^2)^2 + 2a^2(y^2 - x^2) = c$.
2. $y = \dfrac{x}{\sqrt{1-x^2}} + c\,e^{-\frac{x}{\sqrt{1-x^2}}}$.

3. $x(x^2y^2 + \cos xy) = c$.
4. $x(e^y + \ln y + 1) = 1$.
5. $y^2 = cx$ (parabolas with vertex at origin).
6. $x^2y^2(y^2 - x^2) = c$.
7. Catenary curves: $y = \pm a\cosh(x/a)$ and horizontal asymptotes $y = \pm a$.
8. (i) $x^2 + 2y^2 = c$; (ii) $\sin y = ce^{-x}$; (iii) For $k \neq 2$: $y^{2-k} - x^{2-k} = c$; for $k = 2$: $y = cx$.
9. $x = e^{y^2/2}$ or $x = e^{-y^2/2}$ ($x^2e^{-y^2} = 1$).
10. $Q_A(t) = 2Q_0e^{-k_At}, Q_B(t) = Q_0e^{-k_Bt}$. After 1 hour, $\dfrac{Q_A(1)}{Q_B(1)} = 2e^{-(k_A-k_B)} = 1.5 = \dfrac{3}{2} \implies e^{k_A-k_B} = \dfrac{4}{3}$. Equal quantities occur when $2e^{-(k_A-k_B)T} = 1 \implies e^{(k_A-k_B)T} = 2 \implies T = \dfrac{\ln 2}{k_A - k_B} = \dfrac{\ln 2}{\ln(4/3)} = \log_{4/3}(2)$ hours.
11. $y = \dfrac{1}{2}\tan 2x \cos^2 x$.
12. $xy\sin\left(\dfrac{y}{x}\right) = \dfrac{\pi}{2}$.
13. $xe^y(\cos y + \sin y) = e^y \sin y + c$.
14. Let $u = x + 2y$. Solution: $y = \ln((x+2y)^2 + 4(x+2y) + 2) - \dfrac{3}{2\sqrt{2}}\ln\left|\dfrac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right| + c$.

§ Answer Key: Page 7

1. Food supply $F(n) = F_0(1.04)^n$, requirement $R(n) = R_0e^{0.03n}$. Given $F_0 = 0.9R_0$. Self-sufficiency requires $F(n) \geq R(n) \implies 0.9(1.04)^n \geq e^{0.03n} \implies n(\ln(1.04) - 0.03) \geq \ln(1/0.9) = \ln 10 - \ln 9 \implies n \geq \dfrac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$.
2. Water volume element $dV = \pi(4 - (2-h)^2)dh = \pi(4h - h^2)dh$. Flow rate $dV/dt = -A_{\text{out}}V(t) = -12 \times 10^{-4} \times 0.6\sqrt{2gh}$. Integrating from $h = 2$ to $h = 0$ gives $T = \dfrac{\mathbf{78} \times \mathbf{10^5}}{\mathbf{135}\sqrt{\mathbf{g}}}$ seconds.
3. $y = (x - 2\tan^{-1}x)(1 + x^2)$.
4. $F'(x) = f(x) \leq cF(x) \implies F'(x) - cF(x) \leq 0 \implies \dfrac{d}{dx}[e^{-cx}F(x)] \leq 0$. Since $F(0) = 0$ and $F(x) \geq 0$, this implies $F(x) = 0$ for all $x \geq 0 \implies f(x) = 0$.
5. **(C)** $y = -f(x) + ce^x f(x)$.
6. **(B)** $y = x^3 + 3x + 1$.
7. **(C)** $f(x) = \dfrac{1}{4x} + \dfrac{3x^3}{4}$.
8. **(A)** $t = 2\ln 18$.
9. **(C)** 3.

§ Answer Key: Page 8

1. **(c)** $x = 1$.
2. **(b)** 2.
3. **(c)** e.
4. **(c)** $g(x) = 2xe^x$.
5. **(b)** $\left[-\dfrac{2}{e}, \infty\right)$.
6. **1**.
7. **(A)** $y = ce^{-\phi(x)} + \phi(x) - 1$.
8. **(C) 60** min.
9. **(C)** Order 2 and degree 1.

§ Answer Key: Page 9

1. **(B)** 2.
2. **(A)** $y = \sin\dfrac{1}{x} - \cos\dfrac{1}{x}$.
3. **(C)** $m = 3/2$.
4. **(A)** $\sin(x^2y^2) = e^{x-1}$.
5. **(B)** $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$.
6. **(A)** $y = 2 - (2 + a^2)e^{\frac{x^2-a^2}{2}}$.
7. **(A)** $\alpha = -1$.
8. **(D)** $xy = 6$.

§ Answer Key: Page 10

1. **(A)** $e^{x^2}(y^2 - 1) + e^{y^2} = C$.
2. **(A), (C)**.
3. **(d)** $-\dfrac{y^2}{x^2}$.
4. **(A)** 1.
5. **(a), (b)** ($m - n = 1$ with $m = 3, n = 2$).
6. (i) **(b)** H/k; (ii) **(a)** $R\left[1 - \dfrac{k}{H}\right]$; (iii) **(c)** $5R\left[2 - \dfrac{11k}{H}\right]$.
7. **4**.

§ Answer Key: Page 11

1. **6**.
2. **0**.
3. **3**.
4. **(a)** $\dfrac{df}{d\theta} + 2f(\theta)\cot\theta = 0$.
5. $t = \dfrac{2\ln 10}{\ln 2}$ hours $\approx$ **6.64** hours.
6. **(A)** $\dfrac{2\cos x}{(1+\sin x)^2} + ce^{-\sin x}\cos x$.
7. **(A)** $y = e^{x(c - x^3\sin x)}$.
8. **(C)** 3.
9. **(C)** $x^2y^2(y^2 - x^2) = c$.

§ Answer Key: Page 12

1. **(D)** $x^2e^y - \dfrac{x^3}{y} - \dfrac{x}{y^3} = c$.
2. **(B)** $-\dfrac{1}{y} + \cos\dfrac{x}{y} = \sin\dfrac{y}{x} + x + c$.
3. **(D)** $cx^4 = y^7(2x^3y - 1)$.
4. **(A)** $y = \dfrac{1}{2}\csc^2\dfrac{x}{2}$.
5. **4**.
6. **(C)** $\dfrac{y}{x} + e^{-xy} = c$.
7. **(D)** $\left(0, \dfrac{e-1}{2}\right)$.

§ Answer Key: Page 13

1. **(A)** $\sin(y/x) = \log x + \dfrac{1}{2}$.
2. **2**.
3. **(B)** $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$.
4. **(A), (C)**.
5. **(B), (C)**.
6. **(A), (B), (D)**.

§ Answer Key: Page 14

1. **(A), (B), (C)**.
2. **(C), (D)**.
3. **(A)**.
4. **(A), (D)**.
5. **[A]** 3.
6. **[A], [C]**.

§ Answer Key: Page 15

1. **(B), (C)**.
2. **(B), (C)**.
3. **(B), (C), (D)**.
4. $0.4 = \dfrac{2}{5}$.
5. **2**.

§ Answer Key: Page 16

1. **(A), (B)**.
2. **(A), (C)**.
3. **(A), (B), (D)**.
4. **(A), (C)**.
5. **8**.

§ Answer Key: Page 17

1. **(C)**.
2. **(C)** $\dfrac{4e^2}{3}$.
3. **16**.
4. Combining terms: $(4x+3y+1)dx + (3x+2y+1)dy = 0 \iff 2x^2+3xy+y^2+x+y = C$, which factors as $(2x+y+1)(x+y) = C + \text{const}$, proving the lines $x+y=0$ and $2x+y+1=0$ are the asymptotes.
5. General solution: $y = c(x+1) - c^3$; Singular solution (envelope): $27(y+1)^2 = 4(x+1)^3$.
6. **(c)** $\dfrac{\pi}{4}$.

§ Answer Key: Page 18

1. **(a)** $I - \dfrac{kT^2}{2}$.
2. **(a)** $\dfrac{3}{2g(e)}$.
3. **(d)** $-\dfrac{15}{17}$.
4. **(b)** 1.

§ Answer Key: Page 19

1. **(b)** $\dfrac{9}{8}$.
2. **(c)** 4.
3. **(b)** 3.
4. **(b)** $\left(1, \dfrac{\pi}{2}\right)$.
5. **(a)** 0.
6. **(a), (b), (c)**.

§ Answer Key: Page 20

1. Curve C_1: $xy = 4$. Director circle C_2: $x^2 + y^2 = 4$. Shortest distance occurs along $y = x$, at point $(\sqrt{2}, \sqrt{2})$ on C_2 and $(2, 2)$ on C_1, giving distance $\sqrt{8} - 2 = \sqrt{8} - 2$, so $p = 8, q = 2 \implies p^2 - q = 64 - 2 = \mathbf{62}$.
2. Maximum value of $f(x)$ is **1**.
3. $xy - \dfrac{1}{xy} + \ln\left|\dfrac{x}{y}\right| = c$.
4. $(x+2)^4\left(1 + \dfrac{2(y-2)}{x+2}\right) = c\, e^{\frac{2(y-2)}{x+2}}$.
5. Proof follows from direct substitution of $y_2 = y_1 z$ into the linear ODE.
6. $\cot\theta = r + 1 + ce^r$.
7. $(y-x)^{-2} = ce^{x^2} - (1 + x^2)$.
8. $\dfrac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\dfrac{x}{y}} = c$.

§ Answer Key: Page 21

1. $\ln\left|\dfrac{x}{y}\right| + \dfrac{xy}{x-y} = c$.
2. $y = c(1 \pm \cos x)$.
3. $y^2 = 2cx + c^2$.
4. $\ln y = cx + c^2$.
5. Parametric: $x = \dfrac{c}{p^2} - \dfrac{1}{p}$, $y = 2\left(\dfrac{c}{p} - 1\right) + \ln p$.
6. $y^2 = 2ax$.
7. General: $y = cx + c - c^2$; Singular: $(x+1)^2 + 4y - 1 = 0$.
8. Clairaut form: $Y = cX + \dfrac{c}{c-1}$; Singular solution: $y^2 = (x \pm 1)^2$.
9. $y = c_1\cos(ax) + c_2\sin(ax)$.
10. $y = \dfrac{1}{2}a(\ln|x|)^2 + c_1\ln|x| + c_2$.
11. $y = -\dfrac{x^2}{4}$.

§ Answer Key: Page 22

1. $y = \sec x$.

2. $f(x) = \ln x + e^x$.

3. $f(x) = \dfrac{1}{x^3} e^{1-1/x}$.

4. $f(x) = e^x - \cos x$.

5. $f(x) = 1$.

6. $x^2 + y^2 - 2\lambda x + k^2 = 0$.

§ Answer Key: Page 23

1. Fick's law: $\dfrac{dC}{dt} = k(0.0040 - C)$. With $C(0) = 0.0025$ and $C(2) = 0.0030$, we find $e^{-2k} = \dfrac{2}{3}$.

(i) $C(4) = 0.0040 - 0.0015\left(\dfrac{2}{3}\right)^2 = \mathbf{0.00333}$ mg/cm^3.

(ii) For $C(t) = 0.0035$: $e^{-kt} = \dfrac{1}{3} \implies t = 2\dfrac{\ln 3}{\ln(1.5)} \approx \mathbf{5.42}$ hours.

2. By Newton's Law of Cooling, $T(t) - 100 = 100\left(\dfrac{1}{2}\right)^t$. At $t = 2$ min, $T(2) = 100 + 25 = \mathbf{125}$°C.

3. Room temperature 30°F. $T(t) - 30 = (T_0 - 30)e^{-kt}$. At $t = 10$, $T - 30 = -30$; at $t = 20$, $T - 30 = -15$. Hence $e^{-10k} = \dfrac{1}{2}$, so $T_0 - 30 = \dfrac{-30}{1/2} = -60 \implies T_0 = \mathbf{-30}$°F.

4. Volume $V(t) = 100 + t$. ODE: $\dfrac{dS}{dt} + \dfrac{4S}{100 + t} = 10$.

(i) $S(t) = 2(100 + t) - \dfrac{190 \times 10^8}{(100 + t)^4}$ kg.

(ii) At $t = 10$ min, $S(10) = 2(110) - \dfrac{190 \times 10^8}{(110)^4} \approx 220 - 129.8 = \mathbf{90.2}$ kg.

§ Answer Key: Page 24

1. Equation of motion: $m\dfrac{dv}{dt} = W - bv^2$.

(i) $v(t) = v_{\text{term}}\left(\dfrac{1 + C_0 e^{-2\alpha t}}{1 - C_0 e^{-2\alpha t}}\right) = 4.87\left(\dfrac{1 + 0.345e^{-4.02t}}{1 - 0.345e^{-4.02t}}\right)$ m/s. The speed does **NOT** increase indefinitely; it asymptotically approaches terminal velocity.

(ii) $v_{\text{term}} = \sqrt{\dfrac{W}{b}} = \sqrt{\dfrac{712}{30}} \approx \mathbf{4.87}$ m/s.

2. Since $\dfrac{dy}{dt} = u \implies y = ut$, and $\dfrac{dx}{dt} = ky(a - y)$:

(i) $\dfrac{dx}{dy} = \dfrac{k}{u}(ay - y^2) \implies x = \dfrac{\mathbf{k}}{\mathbf{6u}}\mathbf{y^2(3a - 2y)}$.

(ii) Total drift when reaching $y = a$: $d = x(a) = \dfrac{\mathbf{ka^3}}{\mathbf{6u}}$.

3. $x(x - y) = y\ln(ce^x - 1)$.

4. Differentiating $z(x)$: $z'(x) = \dfrac{y(x)}{1 + x^2} \implies y(x) = (1 + x^2)z'(x)$. Substituting into the equation yields $(1 + x^2)z' + 2xz = 3x^2 + 2x + 1 \implies \dfrac{d}{dx}[(1 + x^2)z] = 3x^2 + 2x + 1$. Integrating: $(1 + x^2)z = x^3 + x^2 + x \implies z(x) = \dfrac{x^3 + x^2 + x}{1 + x^2}$. Then $y(x) = \mathbf{1 + x^2} + \dfrac{\mathbf{2x}}{\mathbf{1 + x^2}}$.

§ Answer Key: Page 25

1. $g(x) = 2xe^x$. The local minimum occurs where $g'(x) = 2e^x(1+x) = 0 \implies x = -1$, with $g(-1) = -2/e$. The area bounded is $\int_{-1}^{0}(-2xe^x)dx = \mathbf{2 - \frac{4}{e}}$ sq. units.
2. Interchanging variables and taking derivatives yields $f(x) = \mathbf{x^3}$.
3. Differentiating $I(x) = \int_0^x e^{zx-z^2}dz$ via Leibniz's rule: $I'(x) = 1 + \int_0^x ze^{zx-z^2}dz$. Using integration by parts on $\int_0^x ze^{zx-z^2}dz$ reveals $I'(x) = \frac{x}{2}I(x) + e^{-x^2/4}$. Multiplying by integrating factor $e^{-x^2/4}$ gives $\frac{d}{dx}\left[e^{-x^2/4}I(x)\right] = e^{-x^2/4} \implies I(x) = \mathbf{e^{x^2/4}\int_0^x e^{-z^2/4}\,dz}$.

§ Answer Key: Page 26

1. Expanding the product rule: $xf'(x) + f(x) \le -kf(x) \implies xf'(x) \le -(k+1)f(x)$. Since $\ln(f(x))$ is defined, $f(x) > 0$. Dividing by $xf(x) > 0$ gives:

$$\frac{f'(x)}{f(x)} \le -\frac{k+1}{x}$$

Integrating both sides from 2 to x:

$$\ln\left(\frac{f(x)}{f(2)}\right) \le -(k+1)\ln\left(\frac{x}{2}\right) = \ln\left[\left(\frac{x}{2}\right)^{-(k+1)}\right]$$

Exponentiating yields $f(x) \le f(2)\cdot 2^{k+1}x^{-(k+1)} = Ax^{-1-k}$, where $A = f(2)\cdot 2^{k+1}$ is independent of x. ■

END OF ADVANCED DIFFERENTIAL EQUATIONS MASTER PROBLEM BANK & ANSWER KEY