

Mathsmerizing

Sequences and Progressions

Problem Sheet: Questions 1–197

Name: _____

Date: _____

Show the necessary steps and reasoning. Select one correct option in each MCQ unless marked multiple-select. Answers to non-proof questions appear after Question 197.

- The sums of the first n terms of two arithmetic series are in the ratio $(7n + 1) : (4n + 27)$. Find the ratio of their n th terms.
- The first and last terms of an A.P. are a and b . There are altogether $(2n + 1)$ terms. A new series is formed by multiplying each of the first $2n$ terms by the next term. Show that the sum of the new series is

$$\frac{(4n^2 - 1)(a^2 + b^2) + (4n^2 + 2)ab}{6n}.$$

- In an A.P. whose first term is a , the sum of the first p terms is zero, where $p > 1$. Show that the sum of the next q terms is

$$-\frac{aq(p + q)}{p - 1}.$$

- The sequence $a_1, a_2, a_3, \dots, a_{98}$ satisfies the relation

$$a_{n+1} = a_n + 1, \quad n = 1, 2, 3, \dots, 97,$$

and has sum 4949. Evaluate $\sum_{k=1}^{49} a_{2k}$.

- The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms. Prove that the resulting sum is the square of an integer.
- If the sum of the first $2n$ terms of the A.P. $2, 5, 8, \dots$ is equal to the sum of the first n terms of the A.P. $57, 59, 61, \dots$, then n equals
 (A) 10 (B) 12 (C) 11 (D) 13
- The coefficient of x^8 in the polynomial $(x - 1)(x - 2)(x - 3) \cdots (x - 10)$ is
 (A) 2640 (B) 1320 (C) 1370 (D) 2740
- (Multiple-select.)
 Three numbers in A.P. with common difference d are removed from the first n natural numbers. The average of the remaining numbers is $\frac{43}{4}$. Then the ordered pair (n, d) can be
 (A) (19, 5) (B) (19, 2) (C) (23, 5) (D) (19, 8)

Common information for Questions 9–11

Four different integers form an increasing A.P. One of these numbers is equal to the sum of the squares of the other three numbers.

9. The product of all four numbers is

(A) -2 (B) 1 (C) 0 (D) 2

10. The sum of all four numbers is

(A) 3 (B) 0 (C) 4 (D) 2

11. The common difference of the four numbers is

(A) 1 (B) 3 (C) 2 (D) -2

Common information for Questions 12–14

The natural numbers are divided into sets as follows:

$$\{1\}, \quad \{2, 3\}, \quad \{4, 5, 6\}, \quad \{7, 8, 9, 10\}, \quad \dots$$

12. The sum of the elements in the 50th set is

(A) 61064 (B) 62525 (C) 57625 (D) 60525

13. The last element in the 2007th set is

(A) 2015028 (B) 2107028 (C) 2510208 (D) 2370128

14. The average of the middle two elements of the 2008th set is λ . Then $[\lambda]$ is (where $[x]$ denotes the greatest integer less than or equal to x)

(A) 2106132 (B) 2016012 (C) 2016032 (D) 2160322

15. If $a_1, a_2, a_3, \dots, a_{2n+1}$ are in A.P. and $a_{n+1} \neq 0$, then

$$\frac{a_{2n+1} - a_1}{a_{2n+1} + a_1} + \frac{a_{2n} - a_2}{a_{2n} + a_2} + \dots + \frac{a_{n+2} - a_n}{a_{n+2} + a_n}$$

is equal to

(A) $\frac{n(n+1)}{2} \cdot \frac{a_2 - a_1}{a_{n+1}}$ (B) $\frac{n(n+1)}{2}$ (C) $(n+1)(a_2 - a_1)$ (D) None of these

16. The 10th common term of the progressions $3, 7, 11, \dots$ and $1, 6, 11, \dots$ is

(A) 191 (B) 193 (C) 211 (D) None of these

17. The largest term common to the sequences $1, 11, 21, 31, \dots$ (up to 100 terms) and $31, 36, 41, 46, \dots$ (up to 100 terms) is

(A) 381 (B) 471 (C) 281 (D) None of these

18. Let $AP(a; d)$ denote the set of all terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If

$$AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d),$$

find $a + d$.

- 19.** In an A.P., let S_r denote the sum of the first r terms. If $S_p = q$ and $S_q = p$, where p and q are distinct positive integers, then S_{p+q} is equal to
- (A) 0 (B) $-(p+q)$ (C) $p+q$ (D) pq

- 20.** The sides of a right-angled triangle are in arithmetic progression. If the triangle has area 24 square units, find the length of its smallest side.

- 21.** Let $l_1, l_2, \dots, l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1 , and let $w_1, w_2, \dots, w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2 , where $d_1 d_2 = 10$.
For each $i = 1, 2, \dots, 100$, let R_i be a rectangle with length l_i , width w_i and area A_i . If $A_{51} - A_{50} = 1000$, find the value of $A_{100} - A_{90}$.

- 22.** Suppose that all the terms of an arithmetic progression are natural numbers. The ratio of the sum of the first seven terms to the sum of the first eleven terms is $6 : 11$, and the seventh term lies strictly between 130 and 140. Find the common difference.

- 23.** Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$ and $S_{2n} \neq 0$, find

$$\frac{S_{4n}}{S_{2n}}.$$

- 24.** Let $a_1, a_2, a_3, \dots, a_{100}$ be an arithmetic progression with $a_1 = 3$, and let

$$S_p = \sum_{i=1}^p a_i, \quad 1 \leq p \leq 100.$$

For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n , find all possible values of a_2 . [IIT 2011]

- 25.** Let $a_1, a_2, a_3, \dots, a_{11}$ be real numbers satisfying

$$a_1 = 15, \quad 27 - 2a_2 > 0, \quad a_k = 2a_{k-1} - a_{k-2} \quad (k = 3, 4, \dots, 11).$$

If

$$\frac{a_1^2 + a_2^2 + \dots + a_{11}^2}{11} = 90,$$

find the value of $\frac{a_1 + a_2 + \dots + a_{11}}{11}$. [2010]

- 26.** If a, b, c are in A.P., then $a^2(b+c) + b^2(c+a) + c^2(a+b)$ is equal to

(A) $\frac{(a+b+c)^3}{8}$ (B) $\frac{2}{9}(a+b+c)^3$ (C) $\frac{3}{10}(a+b+c)^3$ (D) $\frac{1}{9}(a+b+c)^3$

Common information for Questions 27–29

[IIT 2007]

Let V_r denote the sum of the first r terms of an arithmetic progression whose first term is r and whose common difference is $(2r - 1)$. Let

$$T_r = V_{r+1} - V_r - 2, \quad Q_r = T_{r+1} - T_r, \quad r = 1, 2, \dots$$

27. The sum $V_1 + V_2 + \dots + V_n$ is

(A) $\frac{1}{12}n(n+1)(3n^2 - n + 1)$

(B) $\frac{1}{12}n(n+1)(3n^2 + n + 2)$

(C) $\frac{1}{2}n(2n^2 - n + 1)$

(D) $\frac{1}{3}(2n^3 - 2n + 3)$

28. T_r is always (select all correct options)

- (A) An odd number (B) An even number (C) A prime number (D) A composite number

29. Which one of the following statements is correct?

(A) $Q_1, Q_2, Q_3, \dots$ are in A.P. with common difference 5.

(B) $Q_1, Q_2, Q_3, \dots$ are in A.P. with common difference 6.

(C) $Q_1, Q_2, Q_3, \dots$ are in A.P. with common difference 11.

(D) $Q_1 = Q_2 = Q_3 = \dots$

Common information for Questions 30–31

The numbers $a_1, a_2, a_3, \dots, a_{1005}$ satisfy

$$\frac{a_1}{a_1 + 1} = \frac{a_2}{a_2 + 3} = \frac{a_3}{a_3 + 5} = \dots = \frac{a_{1005}}{a_{1005} + 2009},$$

and $a_1 + a_2 + \dots + a_{1005} = 2010$.

30. The sequence is a/an

- (A) A.P. (B) G.P. (C) A.G.P. (D) H.P.

31. The 21st term of the sequence is

- (A) $\frac{86}{1005}$ (B) $\frac{83}{1005}$ (C) $\frac{82}{1005}$ (D) $\frac{79}{1005}$

32. The sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals. Determine whether bc^2 , ca^2 and ab^2 , in this order, are in A.P., G.P. or H.P.

33. If

$$(x^2 + x + 1) + (x^2 + 2x + 3) + (x^2 + 3x + 5) + \dots + (x^2 + 20x + 39) = 4500,$$

then x can be (select all correct options)

- (A) 10 (B) -10 (C) 20.5 (D) -20.5

- 34.** Let $a_1, a_2, a_3, \dots$ be real numbers in arithmetic progression such that $a_1 = 15$ and a_2 is an integer. Given

$$\sum_{r=1}^{10} a_r^2 = 1185,$$

let $S_n = \sum_{r=1}^n a_r$. If N is the largest positive integer n for which $S_n \geq S_{n-1}$ (with $S_0 = 0$), find $N - 10$.

- 35.** (*Multiple-select.*)

If $a_1, a_2, a_3, \dots$ are in A.P. with common difference d , and

$$b_k = a_k + a_{k+1} + \dots + a_{k+n-1}, \quad k = 1, 2, 3, \dots,$$

which of the following statements are correct?

(A) $\sum_{k=1}^n b_k = n^2 a_n$

(B) $\sum_{k=1}^n b_k = (n+1)^2 a_n$

(C) $b_k = \frac{n}{2}[a_n + a_1 + 2d(k-1)]$

(D) $\sum_{k=1}^n b_k = n(n+1)a_n$

- 36.** If $\sum_{j=1}^{21} a_j = 693$, where $a_1, a_2, \dots, a_{21}$ are in A.P., and

$$\sum_{i=0}^{10} a_{2i+1} = 5k + l,$$

where k, l are integers with $0 \leq l < 5$, find $\frac{k+l}{25}$.

- 37.** Balls are arranged in rows to form an equilateral triangle. The first row contains one ball, the second row two balls, and so on. If 669 more balls are added, all the balls can be arranged in a square, with each side containing 8 fewer balls than each side of the triangle did. Match Column I with Column II. A value may be used more than once.

Column I

Column II

(A) Initial number of balls

(P) 55

(B) Number of rows in the equilateral triangle

(Q) 1540

(C) Number of balls along a side of the triangle

(R) 20

(D) Number of balls along a side of the square

(S) 47

- 38.** A child was asked to add the first few natural numbers, $1 + 2 + 3 + \dots$, for as long as his patience permitted. When he stopped, he gave the sum as 575. On being told that the result was wrong, he discovered that he had missed exactly one number during addition. The number he missed was
- (A) 10 (B) 18 (C) 20 (D) None of the above

- 39.** In the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots$, where n consecutive terms have the value n , the 150th term is
- (A) 17 (B) 16 (C) 18 (D) None of these

40. ABC is a right-angled triangle with $\angle B = 90^\circ$ and $BC = a$. Points $L_1, L_2, \dots, L_n$ on AB divide it into $n + 1$ equal parts. The segments $L_1M_1, L_2M_2, \dots, L_nM_n$ are parallel to BC , with $M_1, M_2, \dots, M_n$ on AC . The sum of their lengths is

- (A) $\frac{a(n+1)}{2}$ (B) $\frac{a(n-1)}{2}$ (C) $\frac{an}{2}$
 (D) Impossible to determine from the given data

41. A computer solved several problems in succession. The time taken to solve each successive problem was a fixed fraction of the time taken for the preceding problem. It took:

- 63.5 minutes to solve all the problems except the first;
- 127 minutes to solve all the problems except the last;
- 31.5 minutes to solve all the problems except the first two.

How many problems were given to the computer?

42. Let S_n be the sum of the first n terms of a G.P. with first term a and common ratio r , where $r \neq \pm 1$. Prove that

$$S_1 + S_3 + S_5 + \dots + S_{2n-1} = \frac{an}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}.$$

43. The sum of the squares of three distinct real numbers in G.P. is S^2 . If their sum is αS , prove that

$$\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3).$$

44. For $0 < \phi < \frac{\pi}{2}$, let

$$x = \sum_{n=0}^{\infty} \cos^{2n} \phi, \quad y = \sum_{n=0}^{\infty} \sin^{2n} \phi, \quad z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi.$$

Prove that:

- (i) $xyz = xy + z$;
 (ii) $xyz = x + y + z$.

45. a, b, c are the first three terms of a geometric progression. If the harmonic mean of a and b is 12, and that of b and c is 36, find the first five terms of the progression.

46. In a finite G.P., the ratio of the sum of the first eleven terms to the sum of the last eleven terms is $\frac{1}{8}$. The ratio of the sum of all terms except the first nine to the sum of all terms except the last nine is 2. Find the number of terms in the G.P.

47. Let $a_1, a_2, \dots, a_n$ be in G.P. with common ratio $r \neq -1$, and let S_m denote the sum of the first m terms. For $2 \leq m \leq n$, prove that the sum of the products of these m terms taken two at a time is

$$\sum_{1 \leq i < j \leq m} a_i a_j = \frac{r}{r+1} S_m S_{m-1}.$$

48. Find the sum of the series

$$7 + 77 + 777 + \dots$$

to n terms.

49. Let S be the sum, P the product and R the sum of the reciprocals of the n nonzero terms of a G.P. If $S \neq 0$, find the value of

$$P^2 \left(\frac{R}{S} \right)^n.$$

50. The sum of an infinite geometric series is 162, and the sum of its first n terms is 160. If the reciprocal of its common ratio is an integer, find all possible values of the common ratio, n , and the corresponding first term of the series.

51. Let $a_1, a_2, \dots$ be positive real numbers in G.P. For each positive integer n , let A_n , G_n and H_n denote, respectively, the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \dots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \dots, G_n$ in terms of $A_1, A_2, \dots, A_n$ and $H_1, H_2, \dots, H_n$.

52. Let α, β be the roots of $x^2 - x + p = 0$, and let γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$, in this order, are in G.P., then the integral values of p and q , respectively, are

- (A) $-2, -32$ (B) $-2, 3$ (C) $-6, 3$ (D) $-6, -32$

53. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, and let $\Delta = b^2 - 4ac$. If

$$\alpha + \beta, \quad \alpha^2 + \beta^2, \quad \alpha^3 + \beta^3$$

are in G.P., then

- (A) $\Delta \neq 0$ (B) $b\Delta = 0$ (C) $c\Delta = 0$ (D) $\Delta = 0$

54. Let a, b, c be positive integers forming an increasing G.P. If $b - a$ is the perfect square of a natural number, and

$$\log_6 a + \log_6 b + \log_6 c = 6,$$

find the value of $a + b + c - 105$.

55. A sequence is defined by grouping consecutive powers of 5 as follows:

$$T_1 = 1, \quad T_2 = 5 + 5^2, \quad T_3 = 5^3 + 5^4 + 5^5, \quad \dots$$

Find T_{10} .

- (A) $\frac{5^{54}(5^{10} - 1)}{4}$ (B) $\frac{5^{45}(5^9 - 1)}{4}$ (C) $\frac{5^{45}(5^{10} - 1)}{4}$ (D) $\frac{5^{44}(5^{10} - 1)}{4}$

56. If a, b, c, d are nonzero real numbers such that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) \leq (ab + bc + cd)^2,$$

then a, b, c, d are in

- (A) A.P. (B) G.P. (C) H.P. (D) None of these

57. If a, b, c, d, p are distinct real numbers such that

$$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0,$$

then a, b, c, d are in

- (A) A.P. (B) G.P. (C) H.P. (D) None of these

58. In the sequence

$$1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, \dots,$$

each power of 2 occurs consecutively as many times as its value. The 1025th term is

- (A) 2^9 (B) 2^{10} (C) 2^{11} (D) 2^8

59. The number of terms common to the first 100 terms of the progression $1, 2, 4, 8, \dots$ and the first 100 terms of the progression $1, 4, 7, 10, \dots$ is

- (A) 6 (B) 4 (C) 5 (D) None of these

60. Let a, b, c be distinct positive numbers, none equal to 1, in G.P. If $\log_c a$, $\log_b c$ and $\log_a b$, in this order, are in A.P., prove that the common difference of this A.P. is $\frac{3}{2}$.

61. The sum to infinity of a decreasing G.P. is equal to the greatest value of

$$f(x) = x^3 + 3x - 9, \quad x \in [-2, 3].$$

The difference between its first two terms is $f'(0)$. The common ratio of the G.P. is

- (A) $-\frac{2}{3}$ (B) $\frac{4}{3}$ (C) $\frac{2}{3}$ (D) $-\frac{4}{3}$

62. Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in G.P. and their arithmetic mean is $b + 2$, find the value of

$$\frac{a^2 + a - 14}{a + 1}.$$

63. Find the number of real solutions in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ of

$$\sin^{-1} \left(\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2} \right)^i \right) = \frac{\pi}{2} - \cos^{-1} \left(\sum_{i=1}^{\infty} \left(-\frac{x}{2} \right)^i - \sum_{i=1}^{\infty} (-x)^i \right).$$

Here $\sin^{-1}$ and $\cos^{-1}$ take their principal values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.

64. Let α, β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. Define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 1), \quad b_1 = 1, \quad b_n = a_{n-1} + a_{n+1} \quad (n \geq 2).$$

Which of the following options are correct? (Select all correct options.)

(A) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(B) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$.

(C) $a_1 + a_2 + \cdots + a_n = a_{n+2} - 1$ for all $n \geq 1$.

(D) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

65. Let q be a real number with $q \neq 1$, and define

$$S_n = 1 + q + q^2 + \cdots + q^n, \quad T_n = 1 + \left(\frac{q+1}{2}\right) + \cdots + \left(\frac{q+1}{2}\right)^n.$$

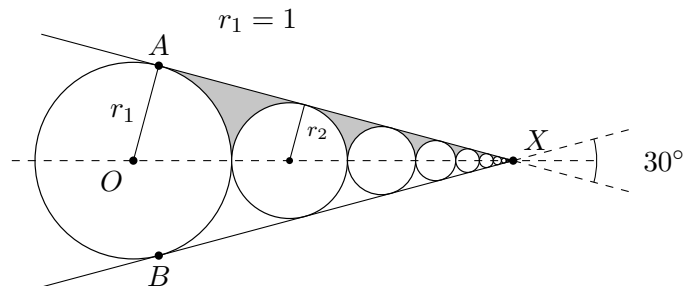
If

$$\binom{101}{1} + \binom{101}{2} S_1 + \cdots + \binom{101}{101} S_{100} = \alpha T_{100},$$

find α .

[JEE Main 2019, 11 January, Shift II]

66. A sequence of mutually tangent circles is inscribed between two lines making an angle of 30° . The circles decrease in size towards the vertex X , and the largest circle has radius $r_1 = 1$. Let S be the sum of the infinitely many shaded areas above the line of centres, as shown. Find $\lfloor S \rfloor$, the greatest integer less than or equal to S .



(A) 0

(B) 1

(C) 2

(D) 3

67. Evaluate the following sums:

(a) $\sum_{r=1}^n \sum_{s=1}^n \delta_{rs} 2^r 3^s$, where $\delta_{rs} = 0$ if $r \neq s$, and $\delta_{rs} = 1$ if $r = s$.

(b) $\sum_{i=1}^n \sum_{j=1}^i \sum_{k=1}^j 1$.

68. If

$$\sqrt{\underbrace{111 \dots 1}_{2n \text{ digits}} - \underbrace{222 \dots 2}_{n \text{ digits}}} = \underbrace{PP \dots P}_{n \text{ digits}},$$

find the digit P .

69. Find the sum of the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \dots,$$

whose terms are the reciprocals of all positive integers of the form $2^u 3^v$, where u, v are nonnegative integers.

70. Let

$$f(x) = \frac{1 - x^{n+1}}{1 - x}, \quad g(x) = 1 - \frac{2}{x} + \frac{3}{x^2} - \dots + (-1)^n \frac{n+1}{x^n}.$$

Which statements about the constant term in $f'(x)g(x)$ are correct? (Select all correct options.)

(A) $\frac{n(n^2 - 1)}{6}$ when n is even.

(B) $\frac{n(n+1)}{2}$ when n is odd.

(C) $-\frac{n}{2}(n+1)$ when n is even.

(D) $-\frac{n(n-1)}{2}$ when n is odd.

71. Let $a_1, a_2, \dots, a_{10}$ be in A.P., and let $h_1, h_2, \dots, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is

(A) 2

(B) 3

(C) 5

(D) 6

72. An A.P. and an H.P. have the same first term, the same last term and the same number of terms. Prove that the product of the r th term from the beginning of one progression and the r th term from the end of the other is independent of r .

73. If a, b, c are in H.P., then the value of

$$\frac{a^3 b^3 + b^3 c^3 + c^3 a^3}{a^2 c^2}$$

is

(A) $9ac - 6b^2$

(B) $3ac - 2b^2$

(C) $9ac - 4b^2$

(D) $9ac - 2b^2$

74. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, find c and all the roots.

75. Let n be a positive integer root of

$$x^2(1 - ac) - x(a^2 + c^2) - (1 + ac) = 0.$$

If n harmonic means $H_1, H_2, \dots, H_n$ are inserted between a and c , prove that

$$H_1 - H_n = ac(a - c).$$

Assume that all the harmonic means are well-defined.

- 76.** Given that α, γ are the roots of $Ax^2 - 4x + 1 = 0$, and β, δ are the roots of $Bx^2 - 6x + 1 = 0$, find A and B such that $\alpha, \beta, \gamma, \delta$, in this order, are in H.P.
- 77.** Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd , in this order, must be in
 (A) None of A.P., G.P., H.P. (B) A.P. (C) G.P. (D) H.P.

- 78.** If x, y, z are nonzero real numbers and

$$4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8zx = 0,$$

then x, y, z are in

- (A) A.P. (B) G.P. (C) H.P. (D) A.G.P.
- 79.** If $a_1, a_2, \dots, a_n$ are in H.P. and

$$f(k) = \left(\sum_{r=1}^n a_r \right) - a_k, \quad k = 1, 2, \dots, n,$$

where $f(k) \neq 0$ for every k , then

$$\frac{a_1}{f(1)}, \quad \frac{a_2}{f(2)}, \quad \frac{a_3}{f(3)}, \quad \dots, \quad \frac{a_n}{f(n)}$$

must be in

- (A) A.P. (B) G.P. (C) H.P. (D) A.G.P.
- 80.** In a triangle ABC , let a, b, c be the sides opposite angles A, B, C , respectively. If

$$\sin^2 \frac{A}{2}, \quad \sin^2 \frac{B}{2}, \quad \sin^2 \frac{C}{2}$$

are in H.P., which of the following must hold?

- (A) a, b, c are in A.P.
 (B) a, b, c are in H.P.
 (C) $a(b+c), b(c+a), c(a+b)$ are in A.P.
 (D) $a(b+c), b(c+a), c(a+b)$ are in H.P.
- 81.** Let $P = (-1, 2)$. A variable line through P meets the x - and y -axes at A and B , respectively. A point Q on the line AB is chosen so that the directed lengths PA, PQ, PB are in H.P. The locus of Q lies on which curve?
 (A) $y = 2x$ (B) $y = -2x$ (C) $y = 2x^2$ (D) $y = -2x^2$

- 82.** If $\cos(x-y), \cos x$ and $\cos(x+y)$ are in H.P., with $\cos y \neq 1$ and $\cos \frac{y}{2} \neq 0$, find all possible values of

$$\cos x \sec \left(\frac{y}{2} \right).$$

- 83.** For projectiles launched with the same speed under uniform gravity, neglecting air resistance, let R_1 and R_2 be the maximum ranges up and down an inclined plane, respectively. Let R be the maximum range on a horizontal plane. Then R_1, R, R_2 , in this order, are in

- (A) Arithmetico-geometric progression (A.G.P.)
- (B) Arithmetic progression (A.P.)
- (C) Geometric progression (G.P.)
- (D) Harmonic progression (H.P.)

- 84.** (*Multiple-select.*)

If a, b, c are three distinct positive numbers in H.P., which of the following statements are correct?

- (A) $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.
- (B) $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.
- (C) $a^5 + c^5 \geq 2b^5$
- (D) $\frac{a-b}{b-c} = \frac{a}{c}$

- 85.** (*Multiple-select.*)

If a, b, c are in H.P., which of the following statements are correct? Assume that all displayed denominators are nonzero.

- (A) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.
- (B) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$
- (C) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P.
- (D) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

- 86.** The harmonic mean of two positive real numbers a and b is 4. If q is a positive real number such that $a, 5, q, b$, in this order, are in A.P., find all possible values of $|q - a|$.

- 87.** For each positive integer t , let

$$f(t) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{t}.$$

Prove that

$$\sum_{k=1}^n (2k+1)f(k) = (n+1)^2 f(n) - \frac{n(n+1)}{2}.$$

88. Let $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and

$$P(n)f(n+2) = P(n)f(n) + Q(n),$$

where P, Q are polynomials of least possible degree and P has leading coefficient 1. Match Column I with Column II.

Column I

Column II

(A) $\sum_{n=1}^m \frac{P(n) - 2}{n}$

(P) $\frac{m(m+1)}{2}$

(B) $\sum_{n=1}^m \frac{Q(n) - 3}{2}$

(Q) $\frac{5m(m+7)}{2}$

(C) $\sum_{n=1}^m \frac{P(n) + Q(n)^2 - 11}{n}$

(R) $\frac{3m(m+7)}{2}$

(D) $\sum_{n=1}^m \frac{Q(n)^2 - P(n) - 7}{n}$

(S) $\frac{m(m+7)}{2}$

89. For an integer $n > 20$, let

$$a = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}, \quad b = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Then $\frac{19a+8b}{2a+b}$ equals

(A) 21

(B) 4

(C) 9

(D) 36

90. Let $a_1 \neq 0$ and $a_n = \frac{n}{a_{n-1}}$ for every integer $n > 1$. Find $a_1 a_2 a_3 \cdots a_{10}$.

91. If $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then for $n \geq 2$, which expressions equal $S_1 + S_2 + \cdots + S_{n-1}$? (Select all correct options.)

(A) $nS_n - n$

(B) $nS_n - 1$

(C) $(n-1)S_{n-1} - n$

(D) $nS_{n-1} - n + 1$

92. Find the sum of the first n terms of

$$1 + 2 \left(1 + \frac{1}{n}\right) + 3 \left(1 + \frac{1}{n}\right)^2 + 4 \left(1 + \frac{1}{n}\right)^3 + \cdots.$$

93. The infinite sum $1 + \frac{4}{7} + \frac{9}{7^2} + \frac{16}{7^3} + \frac{25}{7^4} + \cdots$ equals

(A) $\frac{27}{14}$

(B) $\frac{21}{13}$

(C) $\frac{49}{27}$

(D) $\frac{256}{147}$

94. The value of $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{k}{2^{n+k}}$ is

(A) 5

(B) 4

(C) 3

(D) 2

95. Let

$$\begin{aligned}x &= 1 + 3a + 6a^2 + 10a^3 + \cdots, \quad |a| < 1, \\y &= 1 + 4b + 10b^2 + 20b^3 + \cdots, \quad |b| < 1.\end{aligned}$$

Find $S = 1 + 3ab + 5(ab)^2 + \cdots$ in terms of x and y .

96. Evaluate

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)}.$$

97. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and

$$a_{n+2} = 2a_{n+1} - a_n + 1 \quad \text{for all integers } n \geq 0.$$

Find the value of $\sum_{n=2}^{\infty} \frac{a_n}{7^n}$.

98. If $f(x)$ is a polynomial of degree n such that

$$1 + f(x) = \frac{f(x-1) + f(x+1)}{2} \quad \text{for all } x \in \mathbb{R},$$

find the possible value of n .

99. Find the value of the infinite series

$$\frac{2}{1^3} + \frac{6}{1^3 + 2^3} + \frac{12}{1^3 + 2^3 + 3^3} + \frac{20}{1^3 + 2^3 + 3^3 + 4^3} + \cdots.$$

- (A) 2 (B) $\frac{1}{2}$ (C) 4 (D) $\frac{1}{4}$

100. Let T_r be the r th term of an A.P. whose first term is $-\frac{1}{2}$ and common difference is 1. Then

$$\sum_{r=1}^n \sqrt{1 + T_r T_{r+1} T_{r+2} T_{r+3}}$$

is equal to

- (A) $\frac{n(n+1)(2n+1)}{6} - \frac{5n}{4}$ (B) $\frac{n(n+1)(2n+1)}{6} - \frac{5n}{4} + \frac{1}{4}$
 (C) $\frac{n(n+1)(2n+1)}{6} - \frac{5n}{4} + \frac{1}{2}$ (D) $\frac{n(n+1)(2n+1)}{12} - \frac{5n}{8} + 1$

101. Let $a, b, c \in \mathbb{R}$. If $f(x) = ax^2 + bx + c$ satisfies $a + b + c = 3$ and

$$f(x+y) = f(x) + f(y) + xy \quad \text{for all } x, y \in \mathbb{R},$$

then $\sum_{n=1}^{10} f(n)$ is equal to

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- (A) 255 (B) 330 (C) 165 (D) 190

102. Let $n \geq 100$ and let $a_1, a_2, \dots, a_n$ be real numbers such that

$$\sqrt{a_1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 2} + \dots + \sqrt{a_n - (n-1)} = \frac{1}{2}(a_1 + a_2 + \dots + a_n) - \frac{n(n-3)}{4}.$$

Find the value of $\sum_{i=1}^{100} a_i$.

103. The expression $\sum_{r=1}^n r^2 - \sum_{m=1}^n \sum_{r=1}^m r$ equals

(A) 0

(B) $\frac{1}{2} \left(\sum_{r=1}^n r^2 + \sum_{r=1}^n r \right)$

(C) $\frac{1}{2} \left(\sum_{r=1}^n r^2 - \sum_{r=1}^n r \right)$

(D) None of these

104. Let $f(n) = \sum_{r=1}^n r^4$. Then $\sum_{r=1}^n (2r-1)^4$ is equal to

(A) $f(2n) - 16f(n)$ for all $n \in \mathbb{N}$

(B) $f(n) - 16f\left(\frac{n-1}{2}\right)$ when n is odd

(C) $f(n) - 16f\left(\frac{n}{2}\right)$ when n is even

(D) None of these

105. Find the n th term and the sum of the first n terms of each sequence:

(i) 1, 5, 13, 29, 61, ...

(ii) 6, 13, 22, 33, ...

106. Find the value of

$$\sec^{-1} \left(\frac{1}{4} \sum_{k=0}^{10} \sec \left(\frac{7\pi}{12} + \frac{k\pi}{2} \right) \sec \left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2} \right) \right)$$

that lies in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4} \right]$.

107. Let $N_r = \overbrace{755 \dots 5}^{r \text{ digits}} 7$ denote the $(r+2)$ -digit number whose first and last digits are 7 and whose remaining r digits are 5. Thus $N_0 = 77$. Consider

$$S = N_0 + N_1 + \dots + N_{98} = 77 + 757 + 7557 + \dots + N_{98}.$$

If $S = \frac{N_{99} + m}{n}$, where m, n are natural numbers less than 3000, find $m + n$.

108. (*Multiple-select.*)

For positive real numbers a, b, c , which of the following inequalities are true?

(A) $a^2 + b^2 + c^2 \geq bc + ca + ab$

(B) $a^3 + b^3 + c^3 \geq 3abc$

(C) $(b+c)(c+a)(a+b) \geq 8abc$

(D) $\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq a + b + c$

109. Find the minimum value of

$$\frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}, \quad x > 0.$$

110. Nesbitt's inequality. Prove that for positive real numbers a, b, c ,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

111. Given that x, y, z are positive real numbers such that $xyz = 32$, the minimum value of

$$x^2 + 4xy + 4y^2 + 2z^2$$

is equal to

(A) 64

(B) 256

(C) 96

(D) 216

112. The minimum value of

$$a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10}, \quad a > 0,$$

is

(A) 6

(B) 7

(C) 8

(D) 9

113. Prove that for every integer $n > 1$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} < n\sqrt{\frac{n+1}{2}}.$$

114. If $a, b, c \in (0, \infty)$, prove that

$$9(a^3 + b^3 + c^3) \geq (a + b + c)^3.$$

115. For positive real numbers a, b, c , prove that

$$\frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2} \geq \frac{a + b + c}{3}.$$

- 116.** Let a, b, c be positive real numbers satisfying $a + b + c = 1$. If abc^2 attains its greatest value $\frac{1}{64}$, then

(A) $a = b = \frac{1}{2}, \quad c = \frac{1}{4}$

(B) $a = b = \frac{1}{4}, \quad c = \frac{1}{2}$

(C) $a = b = c = \frac{1}{3}$

(D) None of these

- 117.** If $x > 0$, the minimum value of

$$x^{1000} + x^{900} + x^{90} + x^6 + \frac{1996}{x}$$

is

(A) 1000

(B) 2000

(C) 1996

(D) 3000

- 118.** (*Multiple-select.*)

In $\triangle ABC$, let a, b, c be the side lengths opposite angles A, B, C , respectively. If

$$81 + 144a^4 + 16b^4 + 9c^4 = 144abc,$$

which of the following statements are true?

(A) $a > b > c$

(B) $A < B < C$

(C) The area of $\triangle ABC$ is $\frac{3\sqrt{3}}{8}$.

(D) $\triangle ABC$ is right-angled.

- 119.** If the equation

$$x^4 - 4x^3 + ax^2 + bx + 1 = 0$$

has four positive roots, counted with multiplicity, find a .

- 120.** If a, b, c are positive real numbers, prove that

$$[(1+a)(1+b)(1+c)]^7 > 7^7 a^4 b^4 c^4.$$

- 121.** Let m be the minimum possible value of

$$\log_3(3^{y_1} + 3^{y_2} + 3^{y_3}),$$

where y_1, y_2, y_3 are real numbers satisfying $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of

$$\log_3 x_1 + \log_3 x_2 + \log_3 x_3,$$

where x_1, x_2, x_3 are positive real numbers satisfying $x_1 + x_2 + x_3 = 9$. Find the value of $\log_2(m^3) + \log_3(M^2)$.

122. Let a, b, c, d be positive real numbers such that $a + b + c + d = 11$. If the maximum value of $a^5 b^3 c^2 d$ is 3750β , find β .

123. A geometric progression and a harmonic progression of positive terms have the same p th, q th and r th terms a, b, c , respectively. Prove that

$$a(b - c) \log a + b(c - a) \log b + c(a - b) \log c = 0.$$

124. Let a, b, c, d be positive real numbers, none equal to 1, and suppose they are in G.P. If

$$a^x = b^y = c^z = d^u = k, \quad k > 0, \quad k \neq 1,$$

where x, y, z, u are real numbers, prove that x, y, z, u are in H.P.

125. Find three numbers a, b, c , each lying between 2 and 18, such that

(i) $a + b + c = 25$;

(ii) $2, a, b$ are consecutive terms of an A.P.;

(iii) $b, c, 18$ are consecutive terms of a G.P.

126. Three means x, y, z are inserted between the same endpoints a and b . The means are chosen separately for the A.P. and H.P. cases below.

(a) If $x + y + z = 15$ when a, x, y, z, b are in A.P., and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$ when a, x, y, z, b are in H.P., find a and b .

(b) If $xyz = \frac{15}{2}$ in the A.P. case and $xyz = \frac{18}{5}$ in the H.P. case, find a and b , assuming that they are positive integers.

127. One arithmetic mean a and two geometric means p, q are inserted between two positive numbers. Prove that

$$p^3 + q^3 = 2apq.$$

128. The digits of a three-digit number are three successive terms of a G.P. Subtracting 792 from the number reverses its digits. If 4 is subtracted from its hundreds digit while the other digits remain unchanged, the resulting digits are successive terms of an A.P. Find the original number.

129. The quadratic equations

$$ax^2 + 2bx + c = 0, \quad a_1x^2 + 2b_1x + c_1 = 0$$

have a common root. Suppose $a_1b_1c_1 \neq 0$ and the ratios

$$\frac{a}{a_1}, \quad \frac{b}{b_1}, \quad \frac{c}{c_1}$$

are in a nonconstant A.P. Prove that a_1, b_1, c_1 are in G.P.

- 130.** Let a, b, c, d, e be real numbers such that a, b, c are in A.P., b, c, d are in G.P., and c, d, e are in H.P.

(i) Prove that a, c, e are in G.P.

(ii) Prove that $e = \frac{(2b - a)^2}{a}$.

(iii) If $a = 2$ and $e = 18$, find all possible values of b, c, d .

- 131.** An A.P., a G.P. and an H.P. have a, b as their first two terms. For a positive integer n , prove that their $(n + 2)$ th terms, in this order, are in G.P. if

$$\frac{b^{2n+2} - a^{2n+2}}{ab(b^{2n} - a^{2n})} = \frac{n+1}{n}.$$

Assume that all displayed denominators and the required H.P. terms are well-defined.

- 132.** If $x = 1$, $y > 1$ and $z > 1$ are in G.P., then

$$\frac{1}{1 + \ln x}, \quad \frac{1}{1 + \ln y}, \quad \frac{1}{1 + \ln z}$$

are in

- (A) A.P. (B) H.P. (C) G.P. (D) None of the above

- 133.** Solve the following equations for $x > 0$ and a positive integer y :

$$\log_2 x + \log_4 x + \log_{16} x + \log_{256} x + \cdots = y,$$

$$\frac{5 + 9 + 13 + \cdots + (4y + 1)}{1 + 3 + 5 + \cdots + (2y - 1)} = 4 \log_4 x.$$

The first series is infinite, with each logarithm base equal to the square of the preceding base.

- 134.** Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then a equals

- (A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (D) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

- 135.** Let a, b be positive real numbers. If a, A_1, A_2, b are in A.P., a, G_1, G_2, b are in G.P., and a, H_1, H_2, b are in H.P., prove that

$$\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a + b)(a + 2b)}{9ab}.$$

- 136.** If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ are in G.P.

- 137.** (Multiple-select.)

For $r > 0$, let $S_r = \sqrt{r + \sqrt{r + \sqrt{r + \cdots}}}$. Which statements are correct?

- (A) S_2, S_6, S_{12}, S_{20} are in A.P.
 (B) S_4, S_9, S_{16} are irrational.
 (C) $(2S_3 - 1)^2, (2S_4 - 1)^2, (2S_5 - 1)^2$ are in A.P.
 (D) S_2, S_{12}, S_{56} are in G.P.

Passage for Questions 138–140

The numbers a, b, c lie between 2 and 18 and satisfy $a + b + c = 25$. The numbers 2, a, b are consecutive terms of an A.P., and $b, c, 18$ are consecutive terms of a G.P.

138. The value of abc is

- (A) 500 (B) 450 (C) 720 (D) 480

139. The roots of $ax^2 + bx + c = 0$ are

- (A) Real and positive (B) Real and negative
(C) Non-real (D) Real and of opposite signs

140. If a, b, c are roots of $x^3 + qx^2 + rx + s = 0$, then r equals

- (A) 184 (B) 196 (C) 224 (D) None of these

141. If a_1, a_2, a_3 are in A.P., a_2, a_3, a_4 are in G.P., and a_3, a_4, a_5 are in H.P., then a_1, a_3, a_5 are in

- (A) A.P. (B) G.P. (C) H.P. (D) None of these

142. Let $a, b > 0$ and let n be a positive integer. Suppose

$$\begin{aligned} a, a_1, a_2, \dots, a_{2n-1}, b & \text{ are in A.P.,} \\ a, b_1, b_2, \dots, b_{2n-1}, b & \text{ are in G.P.,} \\ a, c_1, c_2, \dots, c_{2n-1}, b & \text{ are in H.P.} \end{aligned}$$

The roots of $a_n x^2 - b_n x + c_n = 0$ are

- (A) Real and unequal (B) Real and equal (C) Non-real (D) None of these

143. If a, b, c are in G.P. and a, p, q are in A.P., such that $2a, b + p, c + q$ are in G.P., which expressions give possible values of the common difference of the A.P.? Assume $a \neq 0$. (Select all correct options.)

- (A) $\sqrt{2}a$ (B) $(\sqrt{2} + 1)(a - b)$
(C) $\sqrt{2}(a + b)$ (D) $(\sqrt{2} - 1)(b - a)$

144. In the sequence $21, 22, 23, \dots, k - 1, k$, both the arithmetic mean and the geometric mean of the first and last terms occur as terms. If k is a three-digit number, how many values can k take?

- (A) 5 (B) 6 (C) 2 (D) 4

145. Let $b_i > 1$ for $i = 1, 2, \dots, 101$. Suppose $\ln b_1, \ln b_2, \dots, \ln b_{101}$ are in A.P. with common difference $\ln 2$. Let $a_1, a_2, \dots, a_{101}$ be in A.P., with $a_1 = b_1$ and $a_{51} = b_{51}$. If

$$t = b_1 + b_2 + \dots + b_{51}, \quad s = a_1 + a_2 + \dots + a_{51},$$

then

- (A) $s > t$ and $a_{101} > b_{101}$ (B) $s > t$ and $a_{101} < b_{101}$
(C) $s < t$ and $a_{101} > b_{101}$ (D) $s < t$ and $a_{101} < b_{101}$

- 146.** Let $a_1, a_2, \dots$ be positive integers in A.P. with common difference 2, and let $b_1, b_2, \dots$ be positive integers in G.P. with common ratio 2. If $a_1 = b_1 = c$, find the number of possible values of c for which

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer n .

- 147.** Let a, b, c be nonzero real numbers in G.P. The equations

$$ax^2 + 2bx + c = 0, \quad dx^2 + 2ex + f = 0$$

have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

- (A) A.P. (B) G.P. (C) H.P. (D) None of these

Passage for Questions 148–150

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Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, let A_n, G_n, H_n be the arithmetic, geometric and harmonic means, respectively, of A_{n-1} and H_{n-1} .

- 148.** Which of the following statements is correct?

- (A) $G_1 > G_2 > G_3 > \dots$
 (B) $G_1 < G_2 < G_3 < \dots$
 (C) $G_1 = G_2 = G_3 = \dots$
 (D) $G_1 < G_3 < G_5 < \dots$ and $G_2 > G_4 > G_6 > \dots$

- 149.** Which of the following statements is correct?

- (A) $A_1 > A_2 > A_3 > \dots$
 (B) $A_1 < A_2 < A_3 < \dots$
 (C) $A_1 > A_3 > A_5 > \dots$ and $A_2 < A_4 < A_6 < \dots$
 (D) $A_1 < A_3 < A_5 < \dots$ and $A_2 > A_4 > A_6 > \dots$

- 150.** Which of the following statements is correct?

- (A) $H_1 > H_2 > H_3 > \dots$
 (B) $H_1 < H_2 < H_3 < \dots$
 (C) $H_1 > H_3 > H_5 > \dots$ and $H_2 < H_4 < H_6 < \dots$
 (D) $H_1 < H_3 < H_5 < \dots$ and $H_2 > H_4 > H_6 > \dots$

- 151.** Find the sum of the first n terms of each series. Also find its sum to infinity where it converges; otherwise, state that it diverges.

(i) $\frac{1}{1 \cdot 4 \cdot 7} + \frac{1}{4 \cdot 7 \cdot 10} + \frac{1}{7 \cdot 10 \cdot 13} + \dots$

(ii) $\sum_{r=1}^n r(r+1)(r+2)(r+3)$

(iii) $\sum_{r=1}^n \frac{1}{4r^2 - 1}$

(iv) $\frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} + \dots$

- 152.** Sum each of the following series to n terms, assuming all denominators are nonzero:

(i) $\frac{1}{x+1} + \frac{2x}{(x+1)(x+2)} + \frac{3x^2}{(x+1)(x+2)(x+3)} + \dots$

(ii) $\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \dots$

- 153.** If $a_1 = 1$ and $a_n = a_{n-1} + \frac{1}{a_{n-1}}$ for every integer $n > 1$, prove that

$$12 < a_{75} < 15.$$

- 154.** Let $\{u_n\}_{n \geq 1}$ be defined by $u_1 = 1$ and

$$u_{n+1} = u_n + \frac{1}{u_n} \quad (n \geq 1).$$

Prove that $u_n \leq \frac{3\sqrt{n}}{2}$ for every positive integer n .

- 155.** Find the sum of the first n terms of

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \dots$$

- 156.** The sum of the first 10 terms of the series

$$S = \frac{7}{2^2 \cdot 5^2} + \frac{13}{5^2 \cdot 8^2} + \frac{19}{8^2 \cdot 11^2} + \dots$$

is

(A) $\frac{255}{1024}$

(B) $\frac{88}{1024}$

(C) $\frac{264}{1024}$

(D) $\frac{85}{1024}$

- 157.** The value of $\sum_{r=1}^{10} \frac{r}{1-3r^2+r^4}$ is

(A) $-\frac{50}{109}$

(B) $-\frac{54}{109}$

(C) $-\frac{55}{111}$

(D) $-\frac{55}{109}$

- 158.** The value of $\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$ is
- (A) $\frac{1}{120}$ (B) $\frac{1}{96}$ (C) $\frac{1}{24}$ (D) $\frac{1}{144}$

- 159.** The value of $\sum_{r=1}^{\infty} \frac{(4r+5)5^{-r}}{r(5r+5)}$ is
- (A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{1}{25}$ (D) $\frac{2}{25}$

Telescoping sums: Questions 160–162

If $a_r = f(r) - f(r-1)$, then

$$S_n = \sum_{r=1}^n a_r = f(n) - f(0).$$

The infinite sum is $\lim_{n \rightarrow \infty} S_n$, provided this limit is finite.

- 160.** The value of $\sum_{r=1}^{\infty} \frac{(4r-1)5^r}{r^2+r}$ is
- (A) 5 (B) -5 (C) 10 (D) None of these

- 161.** With all angles measured in degrees, the value of

$$(\cos 1^\circ - 1) \sum_{n=1}^{50} \sin n^\circ + \sin 1^\circ \sum_{n=1}^{50} \cos n^\circ$$

is

- (A) $\sin 50^\circ$ (B) $\sin 50^\circ - \sin 1^\circ$ (C) $\sin 51^\circ - \sin 1^\circ$ (D) $\frac{1}{2}$

- 162.** If the sum of the first 50 terms of

$$3(3!) + 4(4!) + 5(5!) + \dots$$

is $a! - b$, where a, b are positive integers with $b < a$, then $a - b$ equals

- (A) 45 (B) 46 (C) 47 (D) 48

- 163.** The sum of the first n terms of $\frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \dots$ can be written as

- (A) $\frac{(n-1)^2}{n^2+2n}$ (B) $\frac{n^2+2n}{(n-1)^2}$ (C) $\frac{n^2+2n}{(n+1)^2}$ (D) $\frac{(n+1)^2}{n^2+2n}$

- 164.** Find the sum of the infinite series

$$\frac{1}{9} + \frac{1}{18} + \frac{1}{30} + \frac{1}{45} + \frac{1}{63} + \dots$$

- (A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$

- 165.** For each positive integer n , let S_n be the sum of the infinite G.P. whose first term is $2n - 1$ and whose common ratio is $\frac{2}{2n+1}$. Then

$$\frac{1}{S_1 S_2 S_3} + \frac{1}{S_2 S_3 S_4} + \frac{1}{S_3 S_4 S_5} + \cdots$$

equals

- (A) $\frac{1}{15}$ (B) $\frac{1}{60}$ (C) $\frac{1}{12}$ (D) $\frac{1}{3}$

- 166.** Let $a_1 = 0$ and let $a_1, a_2, \dots, a_n$ be real numbers such that

$$|a_i| = |a_{i-1} + 1| \quad (2 \leq i \leq n).$$

If their arithmetic mean is A , which statement is always true?

- (A) $A < -\frac{1}{2}$ (B) $A < -1$ (C) $A \geq -\frac{1}{2}$ (D) $A = -\frac{1}{2}$

- 167.** For $a > 0$, $x \neq 0$ and a positive integer n , assume $a + rx > 0$ for $r = 1, \dots, n$. Which expressions equal

$$\sum_{r=1}^n \frac{1}{\sqrt{a+rx} + \sqrt{a+(r-1)x}}?$$

Select all correct options.

- (A) $\frac{n}{\sqrt{a} + \sqrt{a+nx}}$ (B) $\frac{\sqrt{a+nx} - \sqrt{a}}{x}$
 (C) $\frac{n(\sqrt{a+nx} - a)}{x}$ (D) None of these

- 168.** If $H_1, H_2, \dots, H_{2n+1}$ are in a nonconstant H.P., then

$$\sum_{i=1}^{2n} (-1)^i \left(\frac{H_i + H_{i+1}}{H_i - H_{i+1}} \right)$$

is equal to

- (A) $2n - 1$ (B) $2n + 1$ (C) $2n$ (D) $2n + 2$

- 169.** Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be defined by

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n-1} + \sqrt{2n+1}}.$$

Find $\left\lfloor \frac{1}{40} \sum_{r=1}^{40} f(r) \right\rfloor$, where $\lfloor t \rfloor$ denotes the greatest integer not exceeding t .

- 170.** The value of

$$\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

is

- (A) $3 - \sqrt{3}$ (B) $2(3 - \sqrt{3})$ (C) $2(\sqrt{3} - 1)$ (D) $2(2 + \sqrt{3})$

171. The value of $\sum_{r=1}^{\infty} \frac{r^3 + (r^2 + 1)^2}{(r^4 + r^2 + 1)(r^2 + r)}$ is

- (A) $\frac{3}{2}$ (B) 1 (C) 2 (D) Infinite

172. (Multiple-select.)

For each positive integer n , define $S_n : (0, \infty) \rightarrow \mathbb{R}$ by

$$S_n(x) = \sum_{k=1}^n \cot^{-1} \left(\frac{1 + k(k+1)x^2}{x} \right).$$

Use the principal values $\cot^{-1} t \in (0, \pi)$ and $\tan^{-1} t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which statements are true?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1} \left(\frac{1 + 11x^2}{10x} \right)$ for all $x > 0$.

(B) $\lim_{n \rightarrow \infty} \cot(S_n(x)) = x$ for all $x > 0$.

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$.

(D) $\tan(S_n(x)) \leq \frac{1}{2}$ for all $n \geq 1$ and $x > 0$.

173. If $S = \sum_{r=1}^{90} \tan^{-1} \left(\frac{2r}{2 + r^2 + r^4} \right)$, find $8190 \cot S$.

174. (Multiple-select.)

Let $a_1, a_2, a_3, \dots$ be an A.P. with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, \dots$ satisfy $T_1 = 3$ and

$$T_{n+1} - T_n = a_n \quad (n \geq 1).$$

Which of the following statements are true?

(A) $T_{20} = 1604$

(B) $\sum_{k=1}^{20} T_k = 10510$

(C) $T_{30} = 3454$

(D) $\sum_{k=1}^{30} T_k = 35610$

175. Let $\{a_n\}_{n=0}^{\infty}$ satisfy $a_0 = a_1 = 0$ and

$$a_{n+2} = 3a_{n+1} - 2a_n + 1 \quad (n \geq 0).$$

Find the value of

$$a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}.$$

- 176.** Consider the sequence $a_1, a_2, a_3, \dots$ with $a_1 = 1$, $a_2 = 2$ and

$$a_{n+2} = \frac{2}{a_{n+1}} + a_n \quad (n \geq 1).$$

If

$$\left(\frac{a_1 + \frac{1}{a_2}}{a_3}\right) \left(\frac{a_2 + \frac{1}{a_3}}{a_4}\right) \left(\frac{a_3 + \frac{1}{a_4}}{a_5}\right) \dots \left(\frac{a_{30} + \frac{1}{a_{31}}}{a_{32}}\right) = 2^\alpha \binom{61}{31},$$

find α .

- 177.** Find the sum

$$\frac{1^4}{1 \cdot 3} + \frac{2^4}{3 \cdot 5} + \frac{3^4}{5 \cdot 7} + \dots + \frac{n^4}{(2n-1)(2n+1)}.$$

- 178.** Let $a_1, a_2, \dots, a_n$ be in A.P. with common difference $d \neq 0$, and assume $\sin a_i \neq 0$ for every i . The value of

$$(\sin d)(\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \dots + \csc a_{n-1} \csc a_n)$$

is

(A) $\sec a_1 - \sec a_n$

(B) $\csc a_1 - \csc a_n$

(C) $\cot a_1 - \cot a_n$

(D) $\tan a_1 - \tan a_n$

- 179.** For a positive integer n , evaluate

$$\sum_{0 \leq i < j < k \leq n} 1,$$

where the sum is over integer triples (i, j, k) .

- 180.** Prove that, for every positive integer n ,

$$\sum_{1 \leq i \leq j < k \leq w \leq n} 1 = \binom{n}{4} + 2 \binom{n}{3} + \binom{n}{2},$$

where the sum is over integer quadruples (i, j, k, w) and $\binom{n}{r} = 0$ when $r > n$.

- 181.** The sum of the products of the ten numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, taken two at a time, is

(A) 165

(B) -55

(C) 55

(D) None of these

- 182.** Let $a_1, a_2, \dots, a_n$ be a G.P. with common ratio $r \neq -1$, and let S_k denote the sum of its first k terms. For $2 \leq m \leq n$, select the expression equal to

$$\sum_{1 \leq i < j \leq m} a_i a_j,$$

and prove the resulting identity.

(A) $r S_{m-1} S_m$

(B) $\frac{r}{1+r} S_{m-1} S_m$

(C) $\frac{S_{m-1} S_m}{1+r}$

(D) $S_{m-1} S_m$

183. If $f(x) = \frac{2^x}{2^x + \sqrt{2}}$, find the value of

$$\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right),$$

where n is a positive integer.

184. (Multiple-select.)

For a positive integer n , let

$$S_n = \sum_{k=1}^{4n} (-1)^{k(k+1)/2} k^2.$$

Which of the following values can S_n take?

- (A) 1056 (B) 1088 (C) 1120 (D) 1332

185. Evaluate the following sum when $x = 2$:

$$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{2^2}{x^4+1} + \cdots + \frac{2^{100}}{x^{2^{100}}+1}.$$

186. For positive integers n, m , define

$$n_m = n(n-m)(n-2m) \cdots (n-km),$$

where k is the greatest nonnegative integer satisfying $n > km$. The value of $\frac{728}{182}$ is

- (A) 4^9 (B) 4^{18} (C) 256 (D) 36

187. It is known that $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$. Then $\sum_{r=1}^{\infty} \frac{1}{r^2}$ equals

- (A) $\frac{\pi^2}{24}$ (B) $\frac{\pi^2}{3}$ (C) $\frac{\pi^2}{6}$ (D) None of these

188. Given that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90},$$

the value of $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$ is

- (A) $\frac{\pi^4}{96}$ (B) $\frac{\pi^4}{45}$ (C) $\frac{89\pi^4}{90}$ (D) None of these

189. Let $f(n) = \left\lfloor \frac{1}{2} + \frac{n}{100} \right\rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x . The value of

$$\sum_{n=1}^{100} f(n)$$
 is

- (A) 50 (B) 51 (C) 1 (D) None of these

190. Let $\{a_n\}$ satisfy

$$a_1 + a_2 + \cdots + a_n = \frac{n^2 + 3n}{(n+1)(n+2)}$$

for every positive integer n . If

$$28 \sum_{k=1}^{10} \frac{1}{a_k} = p_1 p_2 p_3 \cdots p_m,$$

where $p_1, p_2, \dots, p_m$ are the first m prime numbers, find m .

191. (*Multiple-select.*)

Let

$$a = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \cdots + \frac{1001^2}{2001}, \quad b = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \cdots + \frac{1001^2}{2003}.$$

Which of the following statements are true?

- (A) $\lfloor a - b \rfloor = 500$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x .
- (B) The integer closest to $a - b$ is 501.
- (C) $a - b = 500.74$
- (D) $a - b = 500.94$

192. Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers defined by $a_0 = 0$ and

$$a_{n+1} = a_n + \sqrt{1 + a_n^2} \quad (n \geq 0).$$

If $\lim_{n \rightarrow \infty} \frac{a_n}{2^{n-1}} = \frac{k}{\pi}$, find k .

193. Consider the sequence of natural numbers $S_0, S_1, S_2, \dots$ defined by

$$S_0 = S_1 = 3, \quad S_n = 3 + S_{n-1}S_{n-2} \quad (n > 1).$$

Which of the following statements is correct?

- (A) S_{1545} is odd and S_{1546} is even.
- (B) S_{1545} is even and S_{1546} is odd.
- (C) Both S_{1545} and S_{1546} are even.
- (D) Both S_{1545} and S_{1546} are odd.

194. Prove that a triangle ABC is equilateral if and only if

$$\tan A + \tan B + \tan C = 3\sqrt{3},$$

where all three tangents are defined.

Common information for Questions 195–197

Let $f(n)$ denote the number of different ways in which a positive integer n can be expressed as a sum of 1's and 2's. The order of the summands matters. For example, $f(4) = 5$, since the five representations are

$$2 + 2, \quad 2 + 1 + 1, \quad 1 + 2 + 1, \quad 1 + 1 + 2, \quad 1 + 1 + 1 + 1.$$

Here, $\mathbb{N} = \{1, 2, 3, \dots\}$.

195. The value of $f(6)$ is

(A) 12

(B) 13

(C) 18

(D) 21

196. The value of $f(f(6))$ is

(A) 400

(B) 350

(C) 377

(D) None of these

197. The function $f : \mathbb{N} \rightarrow \mathbb{N}$ is

(A) One-to-one and onto

(B) One-to-one and into (not onto)

(C) Many-to-one and onto

(D) Many-to-one and into (not onto)

Answer Key

Answers to non-proof questions. Proof-only questions are omitted; for mixed questions, only the non-proof parts are answered.

Q.	Answer
1	$\frac{14n - 6}{8n + 23}$
4	2499
6	C (11)
7	B (1320)
8	A, B
9	C (0)
10	D (2)
11	A (1)
12	B (62525)
13	A (2015028)
14	C (2016032)
15	A
16	A (191)
17	D; the largest common term is 521.
18	157
19	B $-(p + q)$
20	6
21	18900
22	9
23	6
24	$a_2 = 3$ or 9 .
25	0
26	B
27	B
28	B, D (always even and composite)
29	B
30	A (A.P.)
31	C (82/1005)
32	A.P.
33	A, D (10, -20.5)

Q.	Answer (continued)
34	6
35	A, C
36	3
37	A–Q, B–P, C–P, D–S
38	C (20)
39	A (17)
40	C ($an/2$)
41	8 problems
45	8, 24, 72, 216, 648
46	38
48	$\frac{7}{81}(10^{n+1} - 10 - 9n)$
49	1
50	$(r, n, a) = (1/81, 1, 160), (1/9, 2, 144), (-1/9, 2, 180), (1/3, 4, 108), (-1/3, 4, 216).$
51	$\left(\prod_{k=1}^n A_k H_k\right)^{1/(2n)}$
52	A ($p = -2, q = -32$)
53	C ($c\Delta = 0$)
54	6
55	C
56	B (G.P.)
57	B (G.P.)
58	B (2^{10})
59	C (5 common terms)
61	C ($2/3$)
62	4
63	2 real solutions
64	B, C, D
65	2^{100}
66	A (0)
67	(a) $\frac{6(6^n - 1)}{5}$; (b) $\frac{n(n+1)(n+2)}{6}$.
68	$P = 3$
69	3
70	B, C
71	D (6)

Q.	Answer (continued)
73	A ($9ac - 6b^2$)
74	$c = 9$; roots $3, -\frac{3}{2}, -\frac{3}{5}$.
76	$A = 3, B = 8$
77	D (H.P.)
78	C (H.P.)
79	C (H.P.)
80	B
81	A ($y = 2x$)
82	$\pm\sqrt{2}$
83	D (H.P.)
84	A, B, C, D
85	A, B, C, D
86	2 or 5
88	A-S, B-P, C-Q, D-R
89	C (9)
90	3840
91	A, D
92	n^2
93	C ($49/27$)
94	D (2)
95	$\frac{1+u}{(1-u)^2}$, where $u = (1 - x^{-1/3})(1 - y^{-1/4})$.
96	$9/32$
97	$7/216$
98	$n = 2$
99	C (4)
100	C
101	B (330)
102	5050
103	C
104	A
105	(i) $a_n = 2^{n+1} - 3$, $S_n = 2^{n+2} - 3n - 4$; (ii) $a_n = n^2 + 4n + 1$, $S_n = \frac{n(2n^2 + 15n + 19)}{6}$.
106	0
107	1219

Q.	Answer (continued)
108	A, B, C, D
109	6
111	C (96)
112	C (8)
116	B
117	B (2000)
118	B, C, D
119	$a = 6$
121	8
122	$\beta = 90$
125	$(a, b, c) = (5, 8, 12)$
126	(a) $(a, b) = (1, 9)$ or $(9, 1)$; (b) $(a, b) = (1, 3)$ or $(3, 1)$.
128	931
130	(iii) $(b, c, d) = (4, 6, 9)$ or $(-2, -6, -18)$.
132	B (H.P.)
133	$x = 2\sqrt{2}$, $y = 3$
134	D
137	A, B, C, D
138	D (480)
139	C (non-real)
140	B (196)
141	B (G.P.)
142	C (non-real)
143	B, D
144	C (2 values: 189, 525)
145	B
146	1 possible value: $c = 12$.
147	A (A.P.)
148	C
149	A
150	B
151	(i) $S_n = \frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}$, $S_\infty = \frac{1}{24}$; (ii) $S_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$, divergent; (iii) $S_n = \frac{n}{2n+1}$, $S_\infty = \frac{1}{2}$; (iv) $S_n = 1 - \frac{2^{\binom{2n+2}{n+1}}}{4^{n+1}}$, $S_\infty = 1$.

Q.	Answer (continued)
152	(i) $1 - \frac{x^n}{\prod_{k=1}^n (x+k)}$; (ii) $1 - \frac{1}{\prod_{k=1}^n (1+a_k)}$.
155	$\frac{1}{2} \left(1 - \frac{1}{n^2 + n + 1} \right)$
156	D (85/1024)
157	D (-55/109)
158	B (1/96)
159	A (1/5)
160	D; the series diverges to $+\infty$.
161	C
162	C (47)
163	C
164	A (1/3)
165	B (1/60)
166	C ($A \geq -1/2$)
167	A, B
168	C ($2n$)
169	9
170	C
171	A ($3/2$)
172	A, B
173	8192
174	B, C
175	528
176	$\alpha = -60$
177	$\frac{n(n+1)(2n+1)}{24} + \frac{n}{16} + \frac{n}{16(2n+1)}$
178	C
179	$\binom{n+1}{3}$
181	B (-55)
182	B (option selection)
183	$2n - 1$
184	A, D
185	$1 - \frac{2^{101}}{2^{2^{101}} - 1}$
186	A (4^9)
187	C ($\pi^2/6$)

Q. Answer (continued)

188 A ($\pi^4/96$)**189** B (51)**190** $m = 6$ **191** A, B**192** $k = 4$ **193** D**195** B (13)**196** C (377)**197** B (one-to-one and not onto)