

Limits

Assignment

1. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q . Let the tangents to the ellipse at P and Q meet at the point R . If $\Delta(h)$ is the area of triangle PQR , $\Delta_1 = \max_{1/2 \leq h \leq 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \leq h \leq 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = \underline{\hspace{2cm}}$.

2. For $a \in \mathbb{R}$, $a \neq -1$,

$$\lim_{n \rightarrow \infty} \frac{1^a + 2^a + \cdots + n^a}{(n+1)^{a-1}[(na+1) + (na+2) + \cdots + (na+n)]} = \frac{1}{60}.$$

Then $a =$

(A) 5

(B) 7

(C) $-\frac{15}{2}$

(D) $-\frac{17}{2}$

3. The largest value of the non-negative integer a for which

$$\lim_{x \rightarrow 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

is $\underline{\hspace{2cm}}$.

4. Let m and n be two positive integers greater than 1. If

$$\lim_{\alpha \rightarrow 0} \frac{e^{\cos(\alpha^n)} - e}{\alpha^m} = -\frac{e}{2},$$

then the value of $\frac{m}{n}$ is $\underline{\hspace{2cm}}$.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous odd function which vanishes exactly at one point, and $f(1) = \frac{1}{2}$. Suppose that

$$F(x) = \int_{-1}^x f(t) dt, \quad G(x) = \int_{-1}^x t |f(f(t))| dt$$

for all $x \in [-1, 2]$. If $\lim_{x \rightarrow 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is $\underline{\hspace{2cm}}$.

6. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then

(A) $\lim_{x \rightarrow 0^+} f'\left(\frac{1}{x}\right) = 1$

(B) $\lim_{x \rightarrow 0^+} xf\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x \rightarrow 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0, 2)$

7. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim_{x \rightarrow 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals $\underline{\hspace{2cm}}$.

8. Let

$$f(x) = \lim_{n \rightarrow \infty} \left\{ \frac{n^n (x+n) \left(x + \frac{n}{2}\right) \cdots \left(x + \frac{n}{n}\right)}{n! \left(x^2 + n^2\right) \left(x^2 + \frac{n^2}{4}\right) \cdots \left(x^2 + \frac{n^2}{n^2}\right)} \right\}^{x/n}$$

for all $x > 0$. Then

- (A) $f\left(\frac{1}{2}\right) \geq f(1)$ (B) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$
 (C) $f'(2) \leq 0$ (D) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

9. Let $f : \mathbb{R} \rightarrow (0, \infty)$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable functions such that f'' and g'' are continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If

$$\lim_{x \rightarrow 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1,$$

then

- (A) f has a local minimum at $x = 2$ (B) f has a local maximum at $x = 2$
 (C) $f''(2) > f(2)$ (D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If

$$g(x) = \int_x^{\pi/2} [f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t f(t)] dt$$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \rightarrow 0} g(x) = \underline{\hspace{2cm}}$.

11. Let $f(x) = \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right)$ for $x \neq 1$. Then

- (A) $\lim_{x \rightarrow 1^-} f(x) = 0$ (B) $\lim_{x \rightarrow 1^-} f(x)$ does not exist
 (C) $\lim_{x \rightarrow 1^+} f(x) = 0$ (D) $\lim_{x \rightarrow 1^+} f(x)$ does not exist

12. For each positive integer n , let

$$y_n = \frac{1}{n} ((n+1)(n+2) \cdots (n+n))^{1/n}.$$

For $x \in \mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x . If $\lim_{n \rightarrow \infty} y_n = L$, then the value of $[L]$ is
 _____.

13. For any positive integer n , define $f_n : (0, \infty) \rightarrow \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{1}{1 + (x+j)(x+j-1)} \right), \quad x \in (0, \infty).$$

Here, $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which of the following statement(s) is (are) true?

(A) $\sum_{j=1}^5 \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} (1 + f'_j(0)) \sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \tan(f_n(x)) = \frac{1}{n}$.

(D) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \sec^2(f_n(x)) = 1$.

14. Let $f : (0, \pi) \rightarrow \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t \rightarrow x} \frac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x \quad \text{for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, which of the following statement(s) is (are) true?

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$ (B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$. (D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

15. For non-negative integers n , let

$$f(n) = \frac{\sum_{k=0}^n \sin\left(\frac{k+1}{n+2}\pi\right) \sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^n \sin^2\left(\frac{k+1}{n+2}\pi\right)}.$$

Assuming $\cos^{-1} x$ takes values in $[0, \pi]$, which of the following options is/are correct?

(A) $f(4) = \frac{\sqrt{3}}{2}$

(B) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$.

(C) $\sin(7 \cos^{-1} f(5)) = 0$

(D) $\lim_{n \rightarrow \infty} f(n) = \frac{1}{2}$

16. For $a \in \mathbb{R}$, $|a| > 1$, let

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt[3]{2} + \cdots + \sqrt[3]{n}}{n^{7/3} \left(\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \cdots + \frac{1}{(an+n)^2} \right)} = 54.$$

Then the possible value(s) of a is/are

- (A) 8 (B) -6
(C) 7 (D) -9

17. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. We say that f has

Property 1: $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{\sqrt{|h|}}$ exists and is finite;

Property 2: $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h^2}$ exists and is finite.

Which of the following options is/are correct?

- (A) $f(x) = x|x|$ has Property 2. (B) $f(x) = \sin x$ has Property 2.
(C) $f(x) = |x|$ has Property 1. (D) $f(x) = x^{2/3}$ has Property 1.

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right-hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{1/x} - e^{-1}}{x^a}$$

is equal to a nonzero real number is _____.

19. Let $f : (-1, 1) \rightarrow \mathbb{R}$ and $g : (-1, 1) \rightarrow (-1, 1)$ be defined by

$$f(x) = |2x - 1| + |2x + 1|, \quad g(x) = x - [x],$$

where $[x]$ denotes the greatest integer less than or equal to x . Let $f \circ g : (-1, 1) \rightarrow \mathbb{R}$ be the composite function defined by $(f \circ g)(x) = f(g(x))$. Suppose c is the number of points in $(-1, 1)$ at which $f \circ g$ is not continuous, and d is the number of points in $(-1, 1)$ at which $f \circ g$ is not differentiable. Then $c + d =$ _____.

20. The value of the limit

$$\lim_{x \rightarrow \pi/2} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2 \sin 2x \sin \frac{3x}{2} + \cos \frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2} \cos 2x + \cos \frac{3x}{2}\right)}$$

is _____.

21. For any positive integer n , let $S_n : (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^n \cot^{-1} \left(\frac{1 + k(k+1)x^2}{x} \right),$$

where $\cot^{-1} x \in (0, \pi)$ and $\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $x \in \mathbb{R}$. Which of the following statement(s) is (are) true?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1} \left(\frac{1 + 11x^2}{10x} \right)$ for all $x > 0$.

(B) $\lim_{n \rightarrow \infty} \cot(S_n(x)) = x$ for all $x > 0$.

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$.

(D) $\tan(S_n(x)) \leq \frac{1}{2}$ for all $n \geq 1$ and $x > 0$.

22. Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = 1$ and $\int_0^{\pi/3} f(t) dt = 0$. Which of the following statement(s) is (are) true?

(A) The equation $f(x) - 3 \cos 3x = 0$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$.

(B) The equation $f(x) - 3 \sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$.

(C) $\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{1 - e^{x^2}} = -1$

(D) $\lim_{x \rightarrow 0} \frac{\sin x \int_0^x f(t) dt}{x^2} = -1$

23. Evaluate $\lim_{x \rightarrow 3\pi/4} \frac{1 + \sqrt[3]{\tan x}}{1 - 2 \cos^2 x}$.

24. (a) Evaluate $\lim_{x \rightarrow 0} \tan^{-1} \left(\frac{a}{x^2} \right)$, where $a \in \mathbb{R}$.

(b) Plot the graph of $f(x) = \lim_{t \rightarrow 0} \left(\frac{2x}{\pi} \tan^{-1} \frac{x}{t^2} \right)$.

25. Find the sum of an infinite geometric series whose first term is the limit of

$$f(x) = \frac{\tan x - \sin x}{\sin^3 x} \quad \text{as } x \rightarrow 0,$$

and whose common ratio is the limit of

$$g(x) = \frac{1 - \sqrt{x}}{(\cos^{-1} x)^2} \quad \text{as } x \rightarrow 1.$$

Use of series expansion or L'Hôpital's rule is not allowed.

26. Evaluate $\lim_{x \rightarrow 0} \frac{8}{x^8} \left[1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4} \right]$.

27. Evaluate $\lim_{x \rightarrow 1} \frac{(\ln(1+x) - \ln 2)(3 \cdot 4^{x-1} - 3x)}{[(7+x)^{1/3} - (1+3x)^{1/2}] \sin(x-1)}.$

28. Let

$$f(x) = \begin{cases} \frac{x}{\sin x}, & x > 0, \\ 2-x, & x \leq 0, \end{cases} \quad g(x) = \begin{cases} x+3, & x < 1, \\ x^2-2x-2, & 1 \leq x < 2, \\ x-5, & x \geq 2. \end{cases}$$

Find the left-hand and right-hand limits of $g(f(x))$ at $x = 0$, and hence find $\lim_{x \rightarrow 0} g(f(x)).$

29. Let $P_n = a^{P_{n-1}} - 1$ for $n = 2, 3, \dots$, and let $P_1 = a^x - 1$, where $a \in \mathbb{R}^+$. Evaluate $\lim_{x \rightarrow 0} \frac{P_n}{x}.$

30. Evaluate $\lim_{x \rightarrow -\infty} \frac{(3x^4 + 2x^2) \sin(1/x) + |x|^3 + 5}{|x|^3 + |x|^2 + |x| + 1}.$

31. If

$$f(x) = \begin{cases} \ln \operatorname{cosec}(\pi x), & 0 < x < 1, \\ \ln \sin(2\pi x), & 1 < x < \frac{3}{2}, \end{cases} \quad g(x) = \frac{2^{f(x)} + 1}{3^{f(x)} + 1},$$

find $\tan^{-1}(g(1^-))$ and $\sec^{-1}(g(1^+)).$

32. At the endpoints and the midpoint of a circular arc AB , tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the two triangles thus formed tends to 4 as the arc AB decreases indefinitely.

33. Evaluate $\lim_{x \rightarrow 0} \left[\frac{(1+x)^{1/x}}{e} \right]^{1/x}.$

34. Evaluate $\lim_{x \rightarrow 0} \left[\sin^2 \left(\frac{\pi}{2-ax} \right) \right]^{\sec^2 \left(\frac{\pi}{2-bx} \right)}.$

35. Evaluate $\lim_{x \rightarrow \infty} x^2 \sin \left(\ln \sqrt{\cos \frac{\pi}{x}} \right).$

36. Evaluate $\lim_{x \rightarrow \infty} \left[\cos \left(2\pi \left(\frac{x}{1+x} \right)^a \right) \right]^{x^2},$ where $a \in \mathbb{R}.$

37.

$$\left(\frac{a_1^{1/x} + a_2^{1/x} + a_3^{1/x} + \dots + a_n^{1/x}}{n} \right)^{nx}, \quad a_1, a_2, \dots, a_n > 0.$$

The source gives this expression without specifying a limit point.

38. Let $f(x) = \frac{\sin^{-1}(1 - \{x\}) \cos^{-1}(1 - \{x\})}{\sqrt{2\{x\}}(1 - \{x\})}$, where $\{x\}$ denotes the fractional part function. Find $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x).$

39. Find the values of a, b, c such that $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2.$

40. Evaluate $\lim_{x \rightarrow a} \frac{1}{(a^2 - x^2)^2} \left(\frac{a^2 + x^2}{ax} - 2 \sin \frac{a\pi}{2} \sin \frac{\pi x}{2} \right)$, where a is an odd integer.

41. Evaluate $\lim_{y \rightarrow 0} \left[\lim_{x \rightarrow \infty} \frac{\exp(x \ln(1 + \frac{ay}{x})) - \exp(x \ln(1 + \frac{by}{x}))}{y} \right]$.

42. If s_n is the sum of the first n terms of the series $\sin x + \sin 2x + \sin 3x + \cdots + \sin nx$, show that

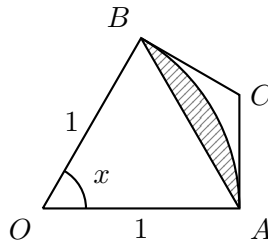
$$\lim_{n \rightarrow \infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = \frac{1}{2} \cot \frac{x}{2}, \quad x \neq 2k\pi, \quad k \in \mathbb{Z}.$$

43. Evaluate $\lim_{x \rightarrow 0} \left[\frac{\ln((1+x)^{1+x})}{x^2} - \frac{1}{x} \right]$.

44. Let $P_n = \frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{4^3 - 1}{4^3 + 1} \cdots \frac{n^3 - 1}{n^3 + 1}$. Evaluate $\lim_{n \rightarrow \infty} P_n$.

45. A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$, as shown in the figure. The point C is the intersection of the two tangent lines at A and B . Let $T(x)$ be the area of triangle ABC , and let $S(x)$ be the area of the shaded region. Compute:

(a) $T(x)$; (b) $S(x)$; (c) $\lim_{x \rightarrow 0} \frac{T(x)}{S(x)}$.



46. Evaluate:

$$(a) \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}}; \quad (b) \lim_{x \rightarrow \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right].$$

47. If $f(n, \theta) = \prod_{r=1}^n \left(1 - \tan^2 \frac{\theta}{2^r} \right)$, compute $\lim_{n \rightarrow \infty} f(n, \theta)$.

48. Evaluate $\lim_{x \rightarrow 0} \frac{2(\tan x - \sin x) - x^3}{x^5}$.

49. Through a point A on a circle, a chord AP is drawn. On the tangent at A , a point T is taken such that $AT = AP$. If TP produced meets the diameter through A at Q , prove that the limiting value of AQ , when P moves up to A , is double the diameter of the circle.

50. Find a and b in each case:

$$(i) \lim_{x \rightarrow \infty} \left[\frac{x^2 + 1}{x + 1} - ax - b \right] = 0; \quad (ii) \lim_{x \rightarrow -\infty} \left[\sqrt{x^2 - x + 1} - ax - b \right] = 0.$$

51. $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ equals

- (A) $-\pi$ (B) π
 (C) $\frac{\pi}{2}$ (D) 1

52. Evaluate $\lim_{x \rightarrow 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$, where $a > 0$.

53. The integer n for which $\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite nonzero number is

- (A) 1 (B) 2
 (C) 3 (D) 4

54. If $\lim_{x \rightarrow 0} \frac{\sin(nx)[(a-n)nx - \tan x]}{x^2} = 0$, where $n > 0$, then a equals

- (A) $\frac{1}{n}$ (B) $n^2 + 1$
 (C) $\frac{n^2 + 1}{n}$ (D) None of these

55. Find $\lim_{n \rightarrow \infty} \left[\frac{2}{\pi}(n+1) \cos^{-1} \left(\frac{1}{n} \right) - n \right]$.

56. If α, β are the roots of $ax^2 + bx + c = 0$, then $\lim_{x \rightarrow \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$ equals

- (A) 0 (B) $\frac{1}{2}(\alpha - \beta)^2$
 (C) $\frac{a^2}{2}(\alpha - \beta)^2$ (D) $-\frac{a^2}{2}(\alpha - \beta)^2$

57. ABC is an isosceles triangle inscribed in a circle of radius r , with $AB = AC$. Let h be the altitude from A to BC , P the perimeter of ABC , and Δ its area. Then $\lim_{h \rightarrow 0} \frac{\Delta}{P^3}$ equals

- (A) $\frac{1}{32r}$ (B) $\frac{1}{64r}$
 (C) $\frac{1}{128r}$ (D) None of these

58. If $[x]$ denotes the greatest integer less than or equal to x , then

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} ([1^3x] + [2^3x] + \cdots + [n^3x])$$

equals

- (A) $x/2$ (B) $x/3$
 (C) $x/6$ (D) $x/4$

59. Let the functions f, g, h be defined as follows:

$$f(x) = \begin{cases} x \sin(1/x), & -1 \leq x \leq 1, x \neq 0, \\ 0, & x = 0, \end{cases}$$

$$g(x) = \begin{cases} x^2 \sin(1/x), & -1 \leq x \leq 1, x \neq 0, \\ 0, & x = 0, \end{cases} \quad h(x) = |x|^3, \quad -1 \leq x \leq 1.$$

Which of these functions are differentiable at $x = 0$?

- (A) f and g only (B) f and h only
(C) g and h only (D) None

60. $\lim_{x \rightarrow \pi/2} \frac{\sin x}{\cos^{-1} \left[\frac{1}{4}(3 \sin x - \sin 3x) \right]}$, where $[\cdot]$ denotes the greatest integer function, is

- (A) $\frac{2}{\pi}$ (B) 1
(C) $\frac{4}{\pi}$ (D) Does not exist

61. The limiting value of $f(x) = \frac{2\sqrt{2} - (\cos x + \sin x)^3}{1 - \sin 2x}$ as $x \rightarrow \frac{\pi}{4}$ is

- (A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$
(C) $3\sqrt{2}$ (D) $\frac{3}{\sqrt{2}}$

62. $\lim_{n \rightarrow \infty} \frac{1^2 n + 2^2(n-1) + 3^2(n-2) + \cdots + n^2 \cdot 1}{1^3 + 2^3 + 3^3 + \cdots + n^3}$ equals

- (A) $\frac{1}{3}$ (B) $\frac{2}{3}$
(C) $\frac{1}{2}$ (D) $\frac{1}{6}$

63. For $f(x) = \lim_{n \rightarrow \infty} \frac{1}{1 + n \sin^2(\pi x)}$, which of the following holds?

- (A) The range of f is a singleton set. (B) f is continuous on $\mathbb{R}$.
(C) f is discontinuous for all $x \in \mathbb{Z}$. (D) f is discontinuous for some $x \in \mathbb{R}$.

64. $\lim_{x \rightarrow 1/\sqrt{2}} \frac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)}$ is

- (A) $\frac{1}{\sqrt{2}}$ (B) $-\frac{1}{\sqrt{2}}$
(C) $\sqrt{2}$ (D) $-\sqrt{2}$

65. $\lim_{x \rightarrow \infty} \frac{\cot^{-1}(\sqrt{x+1} - \sqrt{x})}{\sec^{-1} \left\{ \left(\frac{2x+1}{x-1} \right)^x \right\}}$ equals

- (A) 1 (B) 0
(C) $\pi/2$ (D) Does not exist

66. Let $f(x) = \frac{\ln(x^2 + e^x)}{\ln(x^4 + e^{2x})}$. If $\lim_{x \rightarrow \infty} f(x) = l$ and $\lim_{x \rightarrow -\infty} f(x) = m$, then

- (A) $l = m$ (B) $l = 2m$
 (C) $2l = m$ (D) $l + m = 0$

67. $\lim_{n \rightarrow \infty} \cos(\pi \sqrt{n^2 + n})$, where n is an integer,

- (A) Equals 1 (B) Equals -1
 (C) Equals 0 (D) Does not exist

68. $\lim_{x \rightarrow 0} \frac{(\sin x - \tan x)^2 - (1 - \cos 2x)^4 + x^5}{7(\tan^{-1} x)^7 + (\sin^{-1} x)^6 + 3 \sin^5 x}$ equals

- (A) 0 (B) $\frac{1}{7}$
 (C) $\frac{1}{3}$ (D) 1

69. The value of $\lim_{x \rightarrow 0} \frac{\tan(\{x\} - 1) \sin\{x\}}{\{x\}(\{x\} - 1)}$, where $\{x\}$ denotes the fractional part function, is

- (A) 1 (B) $\tan 1$
 (C) $\sin 1$ (D) Does not exist

70. The value of $\lim_{x \rightarrow 0} (\cos ax)^{\operatorname{cosec}^2 bx}$ is

- (A) e^{-8b^2/a^2} (B) e^{-8a^2/b^2}
 (C) $e^{-a^2/(2b^2)}$ (D) $e^{-b^2/(2a^2)}$

71. $\lim_{x \rightarrow \infty} \frac{2 + 2x + \sin 2x}{(2x + \sin 2x)e^{\sin x}}$ is

- (A) 0 (B) 1
 (C) -1 (D) Does not exist

72. Let

$$f(x) = \begin{cases} \frac{\tan^2\{x\}}{x^2 - [x]^2}, & x > 0, \\ 1, & x = 0, \\ \sqrt{\{x\} \cot\{x\}}, & x < 0, \end{cases}$$

where $[x]$ is the step-up function and $\{x\}$ is the fractional part function. Then

- (A) $\lim_{x \rightarrow 0^+} f(x) = 1$ (B) $\lim_{x \rightarrow 0^-} f(x) = 1$
 (C) $\cot^{-1} \left(\lim_{x \rightarrow 0^-} f(x) \right)^2 = 1$ (D) f is continuous at $x = 1$.

73. $\lim_{x \rightarrow 0} \frac{\sin(6x^2)}{\ln \cos(2x^2 - x)} =$

- (A) 12 (B) -12
 (C) 6 (D) -6

74. $\lim_{x \rightarrow \pi/2} \left[\frac{x - \pi/2}{\cos x} \right]$, where $[\cdot]$ denotes the greatest integer function, is

- (A) -1 (B) 0
 (C) -2 (D) Does not exist

75. If

$$f(x) = \begin{cases} x^2 + 2, & x > 2, \\ 1 - x, & x < 2, \end{cases} \quad g(x) = \begin{cases} 2x, & x > 1, \\ 3 - x, & x \leq 1, \end{cases}$$

evaluate $\lim_{x \rightarrow 1} f(g(x))$.

76. Evaluate $\lim_{n \rightarrow \infty} \frac{[1 \cdot 2x] + [2 \cdot 3x] + \cdots + [n(n+1)x]}{n^3}$, where $[\cdot]$ denotes the greatest integer function.

77. If

$$f(x) = \begin{cases} x - 1, & x > 1, \\ 2x^2 - 2, & x < 1, \end{cases} \quad g(x) = \begin{cases} x + 1, & x > 0, \\ -x^2 + 1, & x \leq 0, \end{cases} \quad h(x) = |x|,$$

then $\lim_{x \rightarrow 0} f(g(h(x)))$ is

- (A) 1 (B) 0
 (C) -1 (D) Does not exist

78. $\lim_{x \rightarrow a^-} \left(\frac{|x|^3}{a} - \left[\frac{x}{a} \right]^3 \right)$, where $a > 0$ and $[x]$ is the greatest integer less than or equal to x , is

- (A) $a^2 + 1$ (B) $a^2 - 1$
 (C) a^2 (D) $-a^2$

79. If α, β are the roots of $ax^2 + bx + c = 0$, then $\lim_{x \rightarrow \alpha} (1 + ax^2 + bx + c)^{1/(x-\alpha)}$ is

- (A) $a(\alpha - \beta)$ (B) $\ln |a(\alpha - \beta)|$
 (C) $e^{a(\alpha-\beta)}$ (D) $e^{a|\alpha-\beta|}$

80. $\lim_{x \rightarrow 0} \left[(1 - e^x) \frac{\sin x}{|x|} \right]$, where $[\cdot]$ represents the greatest integer function, is

- (A) -1 (B) 1
 (C) 0 (D) Does not exist

81. $\lim_{x \rightarrow 0^+} \log_{\sin(x/2)} \sin x$ equals

- (A) 1 (B) 0
 (C) 4 (D) $\frac{1}{4}$

82. If $l = \lim_{x \rightarrow \infty} (\sin \sqrt{x+1} - \sin \sqrt{x})$ and $m = \lim_{x \rightarrow -\infty} [\sin \sqrt{x+1} - \sin \sqrt{x}]$, where $[\cdot]$ denotes the greatest integer function, then

- (A) $l = m = 0$ (B) $l = 0$; m is undefined.
 (C) l, m both do not exist. (D) $l = 0$, $m \neq 0$ (although m exists).

83. Given a real-valued function

$$f(x) = \begin{cases} \frac{\tan^2[x]}{x^2 - [x]^2}, & x > 0, \\ 1, & x = 0, \\ \sqrt{\{x\} \cot\{x\}}, & x < 0, \end{cases}$$

where $[x]$ is the integral part and $\{x\}$ is the fractional part of x , then

- (A) $\lim_{x \rightarrow 0} f(x) = 1$ (B) $\lim_{x \rightarrow 0^-} f(x) = \cot 1$
 (C) $\cot^{-1}\left(\lim_{x \rightarrow 0^-} f(x)\right)^2 = 1$ (D) f is continuous at $x = 0$.

84. The limit $\lim_{\theta \rightarrow 0} \left(\left[\frac{n \sin \theta}{\theta} \right] + \left[\frac{n \tan \theta}{\theta} \right] \right)$, where $[x]$ is the greatest integer function and $n \in \mathbb{N}$, is

- (A) $2n$ (B) $2n + 1$
 (C) $2n - 1$ (D) Does not exist

85. $\lim_{x \rightarrow 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2}$ equals

- (A) $1/2$ (B) $-1/2$
 (C) $3/2$ (D) 1

86. $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + \cdots + \frac{1}{\sqrt{n^2 + 2n}} \right)$ equals

- (A) 1 (B) $1/2$
 (C) 0 (D) 2

87. $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$ equals

- (A) $1/5$ (B) $1/6$
 (C) $1/4$ (D) $1/2$

88. $\lim_{x \rightarrow \infty} \frac{e^x \left((2^{x^n})^{1/e^x} - (3^{x^n})^{1/e^x} \right)}{x^n}$, where $n \in \mathbb{N}$, equals

- (A) 0 (B) $\ln(2/3)$
 (C) $\ln(3/2)$ (D) None of these

89. Show that

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} \left(1 \sum_{k=1}^n k + 3 \sum_{k=1}^{n-1} k + 5 \sum_{k=1}^{n-2} k + \cdots + (2n-1) \cdot 1 \right) = \frac{1}{12}.$$

90. Let $f(x) = \frac{\sin^{-1}(1 - \{x\}) \cos^{-1}(1 - \{x\})}{\sqrt{2\{x\}}(1 - \{x\})}$. Find $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$, where $\{x\}$ denotes the fractional part of x .

91. Let $f(x) = \lim_{m \rightarrow \infty} \left\{ \lim_{n \rightarrow \infty} \cos^{2m}(n!\pi x) \right\}$, where $x \in \mathbb{R}$. Prove that

$$f(x) = \begin{cases} 1, & x \text{ is rational,} \\ 0, & x \text{ is irrational.} \end{cases}$$

92. Evaluate

$$\lim_{x \rightarrow 0^+} \left\{ \lim_{n \rightarrow \infty} \frac{[1^2(\sin x)^x] + [2^2(\sin x)^x] + \cdots + [n^2(\sin x)^x]}{n^3} \right\},$$

where $[\cdot]$ denotes the greatest integer function.

93. Consider the functions

$$f(x) = \lim_{n \rightarrow \infty} \left(\cos \frac{x}{\sqrt{n}} \right)^n, \quad g(x) = -x^{4b},$$

where $b = \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1})$.

(i) $f(x)$ is

(A) e^{-x^2} (B) $e^{-x^2/2}$

(C) e^{x^2} (D) $e^{x^2/2}$

(ii) $g(x)$ is

(A) $-x^2$ (B) x^2

(C) x^4 (D) $-x^4$

(iii) The number of solutions of $f(x) + g(x) = 0$ is

(A) 2 (B) 4

(C) 0 (D) 1

94. For $x > 0$, $\lim_{x \rightarrow 0} ((\sin x)^{1/x} + (1/x)^{\sin x})$ is

(A) 0 (B) -1

(C) 1 (D) 2

95. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be differentiable on $(-1, 1)$. It is given that $f'(0) = \lim_{n \rightarrow \infty} nf\left(\frac{1}{n}\right)$ and $f(0) = 0$. Find

$$\lim_{n \rightarrow \infty} \left[\frac{2}{\pi} (1+n) \cos^{-1}\left(\frac{1}{n}\right) - n \right],$$

given that $\left| \cos^{-1}\left(\frac{1}{n}\right) \right| \leq \frac{\pi}{2}$.

96. $\lim_{x \rightarrow \pi/2} \frac{(1 - \tan \frac{x}{2})(1 - \sin x)}{(1 + \tan \frac{x}{2})(\pi - 2x)^3}$ is

(A) $1/16$ (B) $-1/16$

(C) $1/32$ (D) $-1/32$

97. $\lim_{x \rightarrow \infty} \frac{\log(x^n) - [x]}{[x]}$, where $n \in \mathbb{N}$ and $[x]$ denotes the greatest integer less than or equal to x ,

(A) Has value -1 (B) Has value 0

(C) Has value 1 (D) Does not exist

98. If

$$f(x) = \begin{cases} \frac{[x]^2 + \sin[x]}{[x]}, & [x] \neq 0, \\ 0, & [x] = 0, \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$ equals

- (A) 1 (B) 0
(C) -1 (D) $\lim_{x \rightarrow 0^+} f(x) = 0$

99. The limit

$$\lim_{x \rightarrow a} \left\{ \left[\left(\frac{a^{1/2} + x^{1/2}}{a^{1/4} - x^{1/4}} \right)^{-1} - \frac{2(ax)^{1/4}}{x^{3/4} - a^{1/4}x^{1/2} + a^{1/2}x^{1/4} - a^{3/4}} \right]^{-1} - (\sqrt{2})^{\log_4 a} \right\}^8$$

is

- (A) a (B) $a^{3/4}$
(C) a^2 (D) None of these

100. Let $f(x) = \lim_{n \rightarrow \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}}$. Then

- (A) f is continuous at $x = 1$.
(B) $\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x)$
(C) $\lim_{x \rightarrow 1^+} f(x) = \sin 1$
(D) $\lim_{x \rightarrow 1^-} f(x)$ does not exist.

101. The value of $\lim_{n \rightarrow \infty} \cos \frac{x}{2} \cos \frac{x}{4} \cdots \cos \frac{x}{2^n}$ is

- (A) 1 (B) $\frac{\sin x}{x}$
(C) $\frac{x}{\sin x}$ (D) None of these

102. Let f be a continuous function on $\mathbb{R}$ such that

$$f\left(\frac{1}{4n}\right) = (\sin e^n) e^{-n^2} + \frac{n^2}{n^2 + 1}.$$

Then $f(0)$ is

- (A) 1 (B) $1/2$
(C) 0 (D) None of these

103. If $\lim_{x \rightarrow a} (f(x)g(x))$ exists for functions f and g , then

- (A) $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist.
(B) $\lim_{x \rightarrow a} f(x)$ exists, but $\lim_{x \rightarrow a} g(x)$ may not exist.
(C) $\lim_{x \rightarrow a} f(x)$ may not exist, but $\lim_{x \rightarrow a} g(x)$ exists.
(D) $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ may not exist.

104. $\lim_{x \rightarrow 0} x^8 \left[\frac{1}{x^3} \right]$, where $[x]$ is the greatest integer function, is

- (A) A nonzero real number (B) A rational number
(C) An integer (D) Zero

105. Let

$$\Delta(x) = \begin{vmatrix} \tan x & \tan(x+h) & \tan(x+2h) \\ \tan(x+2h) & \tan x & \tan(x+h) \\ \tan(x+h) & \tan(x+2h) & \tan x \end{vmatrix}.$$

Find $\lim_{h \rightarrow 0} \frac{\sqrt{3} \Delta(\pi/3)}{h^2}$.

106. Match the functions in Column I with the statements in Column II.

Column I

Column II

- (A) $f(x) = x \operatorname{sgn}(x-1)$ (P) $\lim_{x \rightarrow 1} f(x)$ does not exist.
(B) $f(x) = \frac{\sin(\sin(\tan(x^2/2)))}{\log \cos 3x}$ (Q) $\lim_{x \rightarrow 0} f(x)$ does not exist.
(C) $f(x) = \frac{\sqrt[3]{1+\tan^{-1} 3x} - \sqrt[3]{1-\sin^{-1} 3x}}{\sqrt{1-\sin^{-1} 2x} - \sqrt{1+\tan^{-1} 2x}}$ (R) $\lim_{x \rightarrow 0} f(x) = -\frac{1}{9}$
(D) $f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}$ (S) $\lim_{x \rightarrow 0} f(x) = -1$

107. **Statement 1.** If $a, b > 0$ and $[x]$ denotes the greatest integer less than or equal to x , then

$$\lim_{x \rightarrow 0^+} \frac{x}{a} \left[\frac{b}{x} \right] = \frac{b}{a}.$$

Statement 2. $\lim_{x \rightarrow \infty} \frac{\{x\}}{x} = 0$, where $\{x\}$ denotes the fractional part of x .

The source does not include response options for this statement pair.

108. Let $L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - x^2/4}{x^4}$, where $a > 0$. If L is finite, then

- (A) $a = 2$ (B) $a = 1$
(C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

109. Evaluate $\lim_{x \rightarrow 1^-} \frac{1 - \sqrt{x}}{(\cos^{-1} x)^2}$.

110. Evaluate $\lim_{x \rightarrow -1^+} \frac{\sin^{-1}(\sqrt{\pi} - \sqrt{\cos^{-1} x})}{\sqrt{1-x^2}}$.

111. Evaluate $\lim_{x \rightarrow \pi/2} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$.

112. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x^2 - \{x\}\{-x\}}{x \cos x - x^2 - \{x\}\{-x\}}$, where $\{\cdot\}$ denotes the fractional part.

113. Evaluate $\lim_{x \rightarrow 0} \frac{x + \ln(\sqrt{1+x^2} - x)}{x^3}$.

114. If $\lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x + 1} - ax - b \right) = 2$, find a and b .

115. Find a, b if $\lim_{x \rightarrow -\infty} (\sqrt{x^2 - x + 1} - ax - b) = 0$.

116. Find a polynomial $f(x)$ of the least degree such that $\lim_{x \rightarrow 0} \left(1 + \frac{x^2 + f(x)}{x^2} \right)^{1/x} = e^2$.

117. Evaluate $\lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\tan(x^2) + (x+1)^n \sin x}{x^2 + (x+1)^n}$.

118. Evaluate $\lim_{n \rightarrow \infty} \prod_{r=3}^n \frac{r^3 - 8}{r^3 + 8}$.

119. Evaluate $\lim_{x \rightarrow 0} \left(\frac{(1 + [x])^{1/\{x\}}}{e} \right)^{1/\{x\}}$, if it exists, where $\{x\}$ denotes the fractional part of x .

120. $\lim_{x \rightarrow 0} \left(\frac{(1+x)^{2/x}}{e^2} \right)^{4/\sin x}$ is

- (A) e^4 (B) e^{-4}
(C) e^8 (D) e^{-8}

121. If $\lim_{x \rightarrow \infty} \frac{3}{x} \left[\frac{x}{4} \right] = \frac{p}{q}$, where $[\cdot]$ denotes the greatest integer function and p, q are relatively prime, then $p + q$ is

- (A) 2 (B) 7
(C) 5 (D) 6

122. $\lim_{x \rightarrow \pi/2} \frac{\sin x}{\cos^{-1} \left[\frac{1}{4} (3 \sin x - \sin 3x) \right]}$, where $[\cdot]$ denotes the greatest integer function, is

- (A) $\frac{2}{\pi}$ (B) 1
(C) $\frac{4}{\pi}$ (D) Does not exist

123. Let f be a continuous function on $\mathbb{R}$ such that

$$f\left(\frac{1}{4^n}\right) = (\sin e^n) e^{-n^2} + \frac{n^2}{n^2 + 1}.$$

Then $f(0) =$

- (A) 1 (B) 0
(C) -1 (D) $\frac{1}{4}$

124. For $n \in \mathbb{N}$, let $f_n(x) = \tan \frac{x}{2} (1 + \sec x)(1 + \sec 2x)(1 + \sec 4x) \cdots (1 + \sec(2^n x))$. Then $\lim_{x \rightarrow 0} \frac{f_n(x)}{2x}$ equals

- (A) 0 (B) 2^n
(C) 2^{n-1} (D) 2^{n+1}

125. The value of $\lim_{x \rightarrow \pi/4} (1 + [x])^{1/\ln(\tan x)}$, where $[\cdot]$ denotes the greatest integer function, is

- (A) 0 (B) 1
(C) e (D) $1/e$

126. $\lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{\frac{3n^3+4}{4n^4-1}}$, where $n \in \mathbb{N}$, equals

- (A) $(1/e)^{3/4}$ (B) $e^{3/4}$
(C) e^{-1} (D) 0

127. If

$$f(x) = \lim_{n \rightarrow \infty} x \left(\frac{3}{2} + [\cos x] \left(\sqrt{n^2 + 1} - \sqrt{n^2 - 3n + 1} \right) \right),$$

where $[y]$ denotes the largest integer less than or equal to y , identify the correct statement(s).

- (A) $\lim_{x \rightarrow 0} f(x) = 0$ (B) $\lim_{x \rightarrow \pi/2} f(x) = \frac{3\pi}{4}$
(C) $f(x) = \frac{3x}{2}$ for all $x \in \left[0, \frac{\pi}{2}\right]$ (D) $f(x) = 0$ for all $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

128. The value of $\lim_{n \rightarrow \infty} \cos^2 \left(\pi \sqrt[3]{n^3 + n^2 + 2n} \right)$, where $n \in \mathbb{N}$, is

- (A) $\frac{1}{3}$ (B) $\frac{1}{2}$
(C) $\frac{1}{4}$ (D) $\frac{1}{9}$

129. If $\alpha, \beta \in \left(-\frac{\pi}{2}, 0\right)$ satisfy

$$(\sin \alpha + \sin \beta) + \frac{\sin \alpha}{\sin \beta} = 0, \quad (\sin \alpha + \sin \beta) \frac{\sin \alpha}{\sin \beta} = -1,$$

and $\lambda = \lim_{n \rightarrow \infty} \frac{1 + (2 \sin \alpha)^{2n}}{(2 \sin \beta)^{2n}}$, then

- (A) $\alpha = -\frac{\pi}{6}$ (B) $\lambda = 2$
(C) $\alpha = -\frac{\pi}{3}$ (D) $\lambda = 1$

130. Let $f(x) = \begin{cases} |x-2| + a^2 - 6a + 9, & x < 2, \\ 5 - 2x, & x \geq 2. \end{cases}$ If $\lim_{x \rightarrow 2} [f(x)]$ exists, the possible value(s) of a is/are (where $[\cdot]$ represents the greatest integer function)

- (A) 2 (B) $\frac{5}{2}$
(C) 3 (D) $\frac{7}{2}$

131. Let $f(x) = \begin{cases} x+3, & -2 < x < 0, \\ 4, & x = 0, \\ 2x+5, & 0 < x < 1. \end{cases}$

(i) $\lim_{x \rightarrow 0^-} f([x - \tan x])$, where $[\cdot]$ denotes the greatest integer function, is

- (A) 2 (B) 4
(C) 5 (D) None of these

(ii) $\lim_{x \rightarrow 0^+} f\left(\left\{\frac{x}{\tan x}\right\}\right)$, where $\{\cdot\}$ denotes the fractional part function, is

- (A) 4 (B) 5
(C) 7 (D) None of these

132. If $\lim_{x \rightarrow 0} \frac{\ln \cot\left(\frac{\pi}{4} - \beta x\right)}{\tan \alpha x} = 1$, then $\frac{\alpha}{\beta} = \underline{\hspace{2cm}}$.

133. If

$$\lim_{x \rightarrow 0} \frac{f(x)}{\sin^2 x} = 8, \quad \lim_{x \rightarrow 0} \frac{g(x)}{2 \cos x - xe^x + x^3 + x - 2} = \lambda,$$

and $\lim_{x \rightarrow 0} (1 + 2f(x))^{1/g(x)} = \frac{1}{e}$, then $\lambda = \underline{\hspace{2cm}}$.

134. If α, β are two distinct real roots of $ax^3 + x - 1 - a = 0$, where $a \neq -1, 0$, neither of which is equal to unity, and

$$\lim_{x \rightarrow 1/\alpha} \frac{(1+a)x^3 - x^2 - a}{(e^{1-\alpha x} - 1)(x-1)} = \frac{al(k\alpha - \beta)}{\alpha},$$

find kl .

135. Find $\lim_{x \rightarrow \alpha^+} \left[\frac{\min(\sin x, \{x\})}{x-1} \right]$, where α is a root of $\sin x + 1 = x$, $[\cdot]$ represents the greatest integer function, and $\{\cdot\}$ represents the fractional part function.

136. Let $f(x) = \cos 2x \cos 4x \cos 6x \cos 8x \cos 10x$. Then $\lim_{x \rightarrow 0} \frac{1 - (f(x))^3}{5 \sin^2 x}$ equals

- (A) 660 (B) 135
(C) 132 (D) 66

137. If

$$\lim_{x \rightarrow 0} \left(\left[\frac{\sin^{-1} x}{x} \right] + \left[\frac{2^2 \sin^{-1} 2x}{x} \right] + \left[\frac{3^2 \sin^{-1} 3x}{x} \right] + \cdots + \left[\frac{n^2 \sin^{-1} nx}{x} \right] \right) = 100,$$

then the value of n is $\underline{\hspace{2cm}}$.

138. Let $f: \mathbb{R} \rightarrow (0, \infty)$ be such that $f(x) + \frac{e^{x+x^2}}{f(x)} \leq e^x + e^{x^2}$ for all $x > 0$. Then $\lim_{x \rightarrow 1} f(x)$ is

- (A) 1 (B) $1/e$
(C) e (D) $2e$

139. Let

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} \left(\frac{px}{n} \sum_{r=1}^n \frac{[r^2 - e^{-x} + r - 1]}{r(r+1)} \right) + \lambda, & x > 0, \\ q, & x = 0, \\ \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\{r^2 + r + e^x - 1\}}{r(r+1)}, & x < 0 \end{cases}$$

be differentiable on $\mathbb{R}$. Here $[y]$ and $\{y\}$ denote the greatest integer and fractional part functions, respectively.

(i) The value of $p + q + \lambda$ is

- (A) -2 (B) 2
(C) 1 (D) 3

(ii) The value of $f'(\ln 2) + f'\left(\ln \frac{1}{2}\right) + f'\left(\ln \frac{3}{2^2}\right) + f'\left(\ln \frac{5}{2^3}\right) + \dots$ is

- (A) 3 (B) 4
(C) 6 (D) 2

(iii) If g is the inverse of f , then $g'\left(\frac{1}{2}\right)$ is

- (A) $\ln 2$ (B) $-\ln 2$
(C) 2 (D) $\frac{1}{2}$

140. The possible value(s) of k for which

$$\lim_{x \rightarrow \infty} \frac{2x^3 - (\tan^{-1} x)^3}{\frac{8}{\pi} x^3 \cot^{-1} |kx| + k^2 x^6 \sin(1/x^3) - 3kx^3} = \frac{1}{2}$$

is/are

- (A) 0 (B) -1
(C) 2 (D) 4

141. Let

$$f(x) = \frac{\left(\frac{\pi}{2} - \cos^{-1}(1 - \{x\}^2)\right) (\cos^{-1}(1 - \{x\}))^2}{2(\{x\} - \{x\}^3)}.$$

If $f(0^+) = p$ and $f(0^-) = q$, find $\frac{p\pi}{q}$. Here $\{k\}$ denotes the fractional part of k .

142. If α, β ($\alpha < \beta$) are the roots of the equation

$$\lim_{t \rightarrow \infty} \cos^{-1} \left[\sin \left(\tan^{-1} \left(\frac{\sqrt{tx}}{\sqrt{tx^2 - 3tx + t - 1 - x}} \right) \right) \right] = \frac{\pi}{6},$$

find $8^\alpha + 2^\beta - \alpha\beta$.

143. Let

$$f(x) = \lim_{n \rightarrow \infty} \ln \left(\sqrt{e^{\cos x} \sqrt{e^{3 \cos x} \sqrt{e^{5 \cos x} \dots \sqrt{e^{(2n+1) \cos x}}}}} \right).$$

If $g(x) = \left\lfloor \frac{1}{3} f(x) \right\rfloor$, find the number of points in $[0, 2\pi]$ where $g(x)$ is discontinuous. Here $[y]$ denotes the greatest integer less than or equal to y .

144. Let $f(x) = \operatorname{sgn}(x^2 - 4x + 4 + k^2)$, $x \in \mathbb{R}$. If f is discontinuous at exactly one point, then

$$\tan^{-1} k + \cos^{-1} k + \operatorname{cosec}^{-1}(2k - 1) = \frac{m\pi}{2},$$

where m is a whole number. Find m .

145. If $S_n = \sum_{r=1}^n \tan^{-1} \left[\frac{2(2r-1)}{4 + r^2(r^2 - 2r + 1)} \right]$, find

$$\lim_{n \rightarrow \infty} \sum_{r=2}^n (\cot(S_{r-1}) - \cot(S_r)).$$

146. Let $\lim_{n \rightarrow \infty} \frac{1 + cn^2}{(2n + 3 + 2 \sin n)^2} = \frac{1}{2}$. If $c \leq \alpha \leq \beta$, where α, β are the roots of $x^2 - 2px + p^2 - 1 = 0$, find the minimum integral value of p .

147. If

$$f(x) = \lim_{n \rightarrow \infty} \frac{(1 - \cos \{1 - \tan(\frac{\pi}{4} - x)\}) (x+1)^n + \lambda \sin((n - \sqrt{n^2 - 8n})x)}{x^2(x+1)^n + x}, \quad x \neq 0,$$

is continuous at $x = 0$, find $f(0) + 2\lambda$.

148. If $f : (0, \infty) \rightarrow \mathbb{N}$ and

$$f(x) = \left[\frac{x^2 + x + 1}{x^2 + 1} \right] + \left[\frac{4x^2 + x + 2}{2x^2 + 1} \right] + \left[\frac{9x^2 + x + 3}{3x^2 + 1} \right] + \cdots + \left[\frac{n^2x^2 + x + n}{nx^2 + 1} \right], \quad n \in \mathbb{N},$$

find

$$\lim_{n \rightarrow \infty} \left[\frac{f(x) - n}{(f(x))^2 - \frac{n^3(n+2)}{4}} \right].$$

Here $[y]$ denotes the greatest integer less than or equal to y .

149. $\lim_{x \rightarrow 0} \left[\min(y^2 - 4y + 11) \frac{\sin x}{x} \right]$, where $[\cdot]$ denotes the greatest integer function, is

(A) 5

(B) 6

(C) 7

(D) Does not exist

150. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a positive increasing function with $\lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)} = 1$. Then $\lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} =$

(A) $\frac{2}{3}$

(B) $\frac{3}{2}$

(C) 3

(D) 1

Answer Key

Answers transcribed from the supplied key and matched to this sheet's question numbers. The supplied answers have not been independently verified. “–” and “Not supplied” mark missing answers; repeated expressions are identified where an answer is reused.

Questions 1–40

No.	Answer	No.	Answer
1	9	21	A, B
2	B, D	22	A, B, C
3	0	23	$-\frac{1}{3}$
4	2	24	(a) –; (b) $f(x) = x $
5	7	25	$\frac{2}{3}$
6	A	26	$\frac{1}{32}$
7	7	27	$-\frac{9}{4}\ln(4/e)$
8	B, C	28	–3
9	A, D	29	$(\ln a)^n$
10	2	30	–2
11	A, D	31	0, 0
12	1	32	Proof question
13	D	33	$e^{-1/2}$
14	B, C, D	34	e^{-a^2/b^2}
15	A, B, C	35	$\frac{\pi^2}{4}$ (as supplied)
16	A, D	36	$e^{-2\pi^2 a^2}$
17	C, D	37	Not supplied
18	1	38	RHL: $\frac{\pi}{2}$; LHL: $\frac{\pi}{2\sqrt{2}}$
19	4	39	$a = c = 1, b = 2$
20	8	40	$\frac{4+\pi^2 a^2}{16a^4}$

Answer Key

Questions 41–80

No.	Answer	No.	Answer
41	$a - b$	61	D
42	Proof question	62	A
43	Not supplied	63	C
44	$\frac{2}{3}$	64	B
45	(a), (b) $-$; (c) $\frac{3}{2}$	65	A
46	(a) 1; (b) $\frac{1}{2}$	66	A
47	$\theta / \tan \theta$	67	C
48	$\frac{1}{4}$	68	C
49	Proof question	69	D
50	(i) $a = 1$, $b = -1$; (ii) $a = -1$, $b = \frac{1}{2}$	70	C
51	B	71	D
52	$\ln a$	72	A, C
53	C	73	B
54	C	74	C
55	$1 - \frac{2}{\pi}$	75	6
56	C	76	$x/3$
57	C	77	B
58	D	78	C
59	Crossed out in supplied key	79	C
60	A	80	A

Answer Key

Questions 81–120

No.	Answer	No.	Answer
81	A	101	B
82	B	102	Not supplied
83	Not supplied	103	D
84	C	104	B, C, D
85	C	105	432
86	D	106	No matching answers supplied
87	B	107	A (response options absent)
88	B	108	A, C
89	Proof question	109	$\frac{1}{4}$
90	RHL: $\frac{\pi}{2}$; LHL: $\frac{\pi}{2\sqrt{2}}$ (same expression as Q38)	110	$\frac{1}{2\sqrt{\pi}}$
91	Proof question	111	1
92	$\frac{1}{3}$	112	Does not exist
93	(i) B; (ii) A; (iii) A	113	$\frac{1}{6}$
94	C	114	Not supplied
95	$1 - \frac{2}{\pi}$ (same limit as Q55)	115	$a = -1$, $b = \frac{1}{2}$ (same expression as Q50(ii))
96	C	116	$f(x) = -x^2 + 2x$ (as written)
97	A	117	Does not exist
98	D	118	$\frac{2}{7}$
99	C	119	Does not exist; see note
100	B	120	B

Q107: The supplied key marks A, but neither source includes the option definitions. Q116: The polynomial is reproduced as handwritten; the exponent on the final term should be confirmed. Q119: The answer file uses $\{x\}$ in the numerator, whereas the question source prints $[x]$.

Answer Key

Questions 121–150

No.	Answer	No.	Answer
121	Not supplied	136	D
122	A (same expression as Q60)	137	4
123	Not supplied	138	C
124	C	139	(i) D; (ii) B; (iii) C
125	B	140	A, B, D
126	A	141	4
127	A, C, D	142	9
128	C	143	Not supplied
129	A, B	144	Not supplied
130	B, C, D	145	2
131	(i) B; (ii) C	146	3
132	2	147	3
133	8	148	1
134	1	149	B
135	0	150	D

Q140: The corrected markings are read as A, B and D; C is crossed out.