Assignment: Cardano's Method for the Cubic

Mathsmerizing · Theory of Equations · Enrichment

Name: ____________________ Date: ____________

Instructions. Show the full pipeline in every problem: depress → classify by $\Delta = \dfrac{q^2}{4} + \dfrac{p^3}{27}$ *before* extracting radicals → split $y = u + v$ with $uv = -\dfrac{p}{3}$ → solve the hidden quadratic → choose *compatible* cube roots → generate all three roots via $\omega = e^{2\pi i/3}$ → verify $\sum x_k = -A$. Exact forms only; decimals may be added as checks. Answers are given at the end — *final values only, no working* — so you can confirm without being led.

Section A · Core problems

A1. Solve $x^3 + 3x^2 + 4x + 1 = 0$ completely by Cardano's method. First show the equation has no rational root, then find the real root exactly and both complex roots in the form $a \pm bi$.

A2. Solve $x^3 - 3x + 1 = 0$ by Cardano's method. Show $\Delta < 0$, write u^3 and v^3 in polar form, choose a compatible pair of cube roots, and express all three roots as cosines. Verify that each root is real even though u and v are not.

A3. **[$\Delta = 0$ complete walk-through]** Depress $x^3 + 6x^2 + 9x + 4 = 0$, compute Δ, and find all roots using the $\Delta = 0$ root formulas $\left(y = \frac{3q}{p}\text{ simple, } y = -\frac{3q}{2p}\text{ double}\right)$. Confirm by factorisation.

A4. **[A denesting identity]** Apply Cardano to $x^3 + 3x - 4 = 0$ and deduce the (surprising) identity

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1.$$

Section B · Practice set

B1. Solve $x^3 - 6x - 9 = 0$ by Cardano's method. The hidden quadratic has integer roots; give the real root and the conjugate pair exactly.

B2. Solve $x^3 - 3x^2 + 9x - 5 = 0$ completely: depress, classify, and express the real root and the complex pair exactly in terms of $\sqrt[3]{2}$ and $\sqrt[3]{4}$.

B3. **[Bombelli's equation, 1572]** Solve $x^3 - 15x - 4 = 0$. Cardano's formula gives $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$; simplify the cube roots exactly, recover the rational root, and find the remaining two roots.

B4. Solve $x^3 - 12x + 8 = 0$. Show $\Delta < 0$ and express all three roots as cosines using the substitution $x = 4\cos\theta$.

B5. Show that

$$\alpha = \sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}}$$

is a root of a cubic $x^3 + px + q = 0$ with integer p, q; find that cubic and hence evaluate α exactly.

B6. Classify and solve $x^3 - 12x + 16 = 0$ using the $\Delta = 0$ root formulas, and confirm your answer by factorisation.

ANSWERS (no solutions — attempt first)

Section A.

A1. With $U = \sqrt[3]{\dfrac{9+\sqrt{93}}{18}}$, $V = \sqrt[3]{\dfrac{9-\sqrt{93}}{18}}$ (real cube roots):

$$x_1 = -1 + U + V \approx -0.3177,$$
$$x_{2,3} = -1 - \frac{U+V}{2} \pm \frac{\sqrt{3}}{2}(U-V)\,i \approx -1.3412 \pm 1.1615\,i.$$

A2. $\Delta = -\dfrac{3}{4} < 0$; $\quad x = 2\cos\dfrac{2\pi}{9},\ 2\cos\dfrac{4\pi}{9},\ 2\cos\dfrac{8\pi}{9} \approx 1.5321,\ 0.3473,\ -1.8794.$

A3. Depressed: $y^3 - 3y + 2 = 0$ with $\Delta = 0$; roots $x = -1$ (double), $x = -4$; indeed $x^3 + 6x^2 + 9x + 4 = (x+1)^2(x+4)$.

A4. $\Delta = 5 > 0$, so the cubic has a unique real root, namely $x = 1$; Cardano's radical expression for that root is $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$, forcing the identity.

Section B.

B1. $x_1 = 3, \qquad x_{2,3} = -\dfrac{3}{2} \pm \dfrac{\sqrt{3}}{2}\,i.$

B2. Depressed ($x = y+1$): $y^3 + 6y + 2 = 0$, $\Delta = 9 > 0$;

$$x_1 = 1 + \sqrt[3]{2} - \sqrt[3]{4} \approx 0.6724, \qquad x_{2,3} = 1 - \frac{\sqrt[3]{2} - \sqrt[3]{4}}{2} \pm \frac{\sqrt{3}}{2}(\sqrt[3]{2} + \sqrt[3]{4})\,i.$$

B3. $\sqrt[3]{2 \pm 11i} = 2 \pm i;\quad x_1 = 4, \qquad x_{2,3} = -2 \pm \sqrt{3}.$

B4. $\Delta = -48 < 0;\quad x = 4\cos\dfrac{2\pi}{9},\ 4\cos\dfrac{4\pi}{9},\ 4\cos\dfrac{8\pi}{9} \approx 3.0642,\ 0.6946,\ -3.7588.$

B5. The cubic is $x^3 + 6x - 20 = 0$; $\quad \alpha = 2$ (equivalently $10 \pm 6\sqrt{3} = (1 \pm \sqrt{3})^3$).

B6. $\Delta = 0$; roots $x = 2$ (double), $x = -4$; $\quad x^3 - 12x + 16 = (x-2)^2(x+4)$.