Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and satisfy

$$f(x+y) = f(x) + f(y) + xy \quad \forall x, y \in \mathbb{R},$$

with $f'(0) = 1$. Find $f(x)$ explicitly.

Answer. $f(x) = x + \dfrac{x^2}{2}$.

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and satisfy

$$f(x+y) = f(x) + f(y) + (e^x - 1)(e^y - 1) \quad \forall x, y \in \mathbb{R},$$

with $f'(0) = 2$. Find $f(x)$ explicitly.

Answer. $f(x) = e^x + x - 1$.

Problem 3

Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and satisfy

$$f(xy) = f(x) + f(y) + (\ln x)(\ln y) \quad \forall x, y > 0,$$

with $f(1) = 0$.

(a) Differentiate with respect to y and set $y = 1$ to derive an ODE.

(b) Solve the ODE.

(c) Show that $f'(1)$ is a free parameter and is not determined by the given conditions.

Answer. $f(x) = \dfrac{(\ln x)^2}{2} + c \ln x, \quad c \in \mathbb{R}$ ($c = f'(1)$ free).

Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and satisfy

$$f(x+y) + f(x-y) = 2f(x) + 2y^2 \quad \forall x, y \in \mathbb{R}.$$

(a) Set $x = 0$ and obtain a constraint on f.

(b) Differentiate twice with respect to y and set $y = 0$.

(c) Find all possible f.

Answer. $f(x) = x^2 + bx + d, \quad b, d \in \mathbb{R}$.

Problem 5

Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and satisfy

$$f(xy) = xf(y) + yf(x) \quad \forall x, y > 0.$$

(a) Show that $f(1) = 0$.

(b) Differentiate with respect to y and set $y = 1$.

(c) Solve the resulting first-order linear ODE.

(d) Verify the solution.

Answer. $f(x) = k\,x \ln x, \quad k \in \mathbb{R}.$

Problem 6

Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and satisfy

$$f(xy) = x^2 f(y) + y^2 f(x) \quad \forall x, y > 0,$$

with $f(e) = e^2$. Find $f(x)$.

Answer. $f(x) = x^2 \ln x.$

Problem 7

Let $n \in \mathbb{N}$ be fixed. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable and satisfy

$$f(xy) = x^n f(y) + y^n f(x) \quad \forall x, y \neq 0.$$

(a) Show that $f(1) = 0$ and $f(-1) = 0$.

(b) Derive the ODE by differentiating with respect to y and setting $y = 1$.

(c) Show that all solutions have the form $f(x) = kx^n \ln |x|$ for some constant $k \in \mathbb{R}$.

Answer. $f(x) = k\,x^n \ln |x|, \quad k \in \mathbb{R}.$

Problem 8

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and satisfy

$$f(x + y) + f(x - y) = 2f(x) \cos y \quad \forall x, y \in \mathbb{R}.$$

Find all such f.

Answer. $f(x) = A \cos x + B \sin x, \quad A, B \in \mathbb{R}.$

Problem 9

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and satisfy

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2 \quad \forall x, y \in \mathbb{R},$$

with $f(0) = 0$, $f'(0) = \alpha \neq 0$. Determine all members of the complete solution family that satisfy this initial condition.

Answer. $f(x) = \alpha x, \ \ \dfrac{\alpha}{k}\sin(kx), \ \ \dfrac{\alpha}{k}\sinh(kx), \quad k > 0.$

Problem 10

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions satisfying

$$f(x+y) = f(x)g(y) \quad \forall x, y \in \mathbb{R}.$$

Assume $f \not\equiv 0$. Show that there exist constants $C \neq 0$ and $\lambda \in \mathbb{R}$ such that $f(x) = Ce^{\lambda x}$, $g(x) = e^{\lambda x}$.

Answer. $f(x) = Ce^{\lambda x}, \ \ g(x) = e^{\lambda x}, \quad C \neq 0, \ \lambda \in \mathbb{R}.$

Problem 11

Let $f : (0, \infty) \to (0, \infty)$ be differentiable and satisfy

$$\frac{f(xy)}{f(x)} = f(y) \quad \forall x, y > 0.$$

Show that $f(x) = x^{\lambda}$ for some $\lambda \in \mathbb{R}$.

Answer. $f(x) = x^{\lambda}, \quad \lambda \in \mathbb{R}.$

Problem 12

Let $f : (0, \infty) \to (0, \infty)$ be differentiable and satisfy

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 \quad \forall x, y > 0.$$

(a) Set $F = \ln f$.

(b) Show that F satisfies Jensen's midpoint equation.

(c) Use differentiability to show that F is affine.

(d) Conclude that $f(x) = Ce^{ax}$ for constants $C > 0$ and $a \in \mathbb{R}$.

(e) State what changes if differentiability is replaced by continuity.

Answer. $f(x) = Ce^{ax}, \quad C > 0, \ a \in \mathbb{R}.$

Problem 13

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy

$$f'(x+y) = f'(x)f'(y) \quad \forall x, y \in \mathbb{R}.$$

Find all such f.

Answer. $f(x) = \dfrac{e^{\lambda x}}{\lambda} + C\ (\lambda \neq 0),\ \ x + C,\ \ C.$

Problem 14

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy

$$f(x+y) + f(x-y) = 2f(x) \quad \forall x, y \in \mathbb{R}.$$

(a) Differentiate with respect to y.

(b) Use the substitution $u = x+y$, $v = x-y$ to show that f' is constant.

(c) Find all possible f.

Answer. $f(x) = mx + c, \quad m, c \in \mathbb{R}.$

Problem 15

Let $f : \mathbb{R} \to [0, \infty)$ be differentiable and satisfy

$$f(x+y) = f(x) + f(y) + 2\sqrt{f(x)}\sqrt{f(y)} \quad \forall x, y \in \mathbb{R}.$$

Find all such f.

Answer. $f \equiv 0$ (unique). *If the domain is restricted to* $[0, \infty)$: the move $y = -x$ is unavailable, the equation reduces to $\sqrt{f(x+y)} = \sqrt{f(x)} + \sqrt{f(y)}$, and the solutions become $f(x) = c^2 x^2,\ c \in \mathbb{R}$.

Problem 16

Let $I \subset \mathbb{R}$ be an interval containing 0. Let $f : I \to \mathbb{R}$ be differentiable and satisfy

$$f(x+y)\big(1 - f(x)f(y)\big) = f(x) + f(y)$$

whenever $x, y, x+y \in I$. Find the local family of solutions.

Answer. $f(x) = \tan(\lambda x), \quad \lambda \in \mathbb{R}$ (local, $|\lambda x| < \pi/2$).

Problem 17

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy

$$f(x) = \sin x + \int_0^x f(t)\,dt \quad \forall x \in \mathbb{R}.$$

Find $f(x)$.

Answer. $f(x) = \dfrac{1}{2}\big(\sin x - \cos x + e^x\big).$

Problem 18

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy

$$f(x) = 1 + \int_0^x (x-t) f(t)\, dt \quad \forall x \in \mathbb{R}.$$

(a) Differentiate once.

(b) Differentiate again.

(c) Solve the resulting second-order ODE.

(d) Verify the solution in the original integral equation.

Answer. $f(x) = \cosh x$.

Problem 19

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period 1, so $f(x+1) = f(x)\ \forall x \in \mathbb{R}$. Define $g(t) = \int_0^t f(x)\, dx$, and let $A = \int_0^1 f(x)\, dx$.

(a) Show that $g(t+1) - g(t) = A$.

(b) Show that for every fixed t, $\lim_{n\to\infty} \frac{g(t+n)}{n} = A$.

(c) Show that for every fixed t, $\lim_{n\to\infty} \frac{g(nt)}{n} = At$.

(d) Deduce that $\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)\, dx = A$.

(e) If additionally $f(x) = \lambda + \int_0^x f(t)\, dt\ \forall x \in \mathbb{R}$, find f and A.

Answer. All four limits equal A (resp. At); the extra equation forces $f \equiv 0,\ A = 0$.

Problem 20

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy

$$f(x+y) = f(x)g(y) + f(y)g(x), \qquad g(x+y) = g(x)g(y) - f(x)f(y),$$

for all $x, y \in \mathbb{R}$. Assume $f(0) = 0$, $g(0) = 1$, $f'(0) = \alpha$, $g'(0) = 0$.

(a) Differentiate both equations with respect to y and set $y = 0$.

(b) Derive a system of ODEs for f and g.

(c) Show that $f(x)^2 + g(x)^2 = 1$.

(d) Derive a second-order ODE for f.

(e) Find f and g.

Answer. $f(x) = \sin(\alpha x),\ \ g(x) = \cos(\alpha x)$.

Problem 21

For each equation below, identify the correct solving model before solving.

(1) Perturbed additive Cauchy: $f(x+y) = f(x) + f(y) + x^2y + xy^2$.

(2) Multiplicative logarithmic Cauchy: $f(xy) = f(x) + f(y) + (\ln x)^2 \ln y$, $x, y > 0$.

(3) Symmetric second-order equation: $f(x+y) + f(x-y) = 2f(x)\cosh y$.

(4) Integral equation: $f(x) = x + \int_0^x (x-t)f(t)\,dt$.

(5) Local tangent equation: $f(x+y)(1 - f(x)f(y)) = f(x) + f(y)$.

For each, state: FE model $\longrightarrow$ ODE obtained $\longrightarrow$ solution family.

Answer. (1) $f(x) = \frac{x^3}{3} + cx$; (2) $f(x) = \frac{(\ln x)^3}{3} + c\ln x$; (3) $f(x) = C\cosh x + D\sinh x$; (4) $f(x) = \sinh x$; (5) $f(x) = \tan(\lambda x)$ or $f \equiv 0$.