1. If $x = 1 + \sqrt[5]{2} + \sqrt[5]{4} + \sqrt[5]{8} + \sqrt[5]{16}$, then the value of $\left(1 + \frac{1}{x}\right)^{30}$ is

(A) 2 (B) 5 (C) 32 (D) 64

$x = 1 + 2^{1/5} + 2^{2/5} + \cdots + 2^{4/5}$ G.P.

$x = \frac{1\,((2^{1/5})^5 - 1)}{2^{1/5} - 1}$ $\quad x = \frac{1}{2^{1/5} - 1}$ $\quad 2^{1/5} = 1 + \frac{1}{x}$

$\left(1 + \frac{1}{x}\right)^5 = 2$ $\quad \left(1 + \frac{1}{x}\right)^{30} = 2^6 = 64$

2. Let j be a number selected at random from $\{1, 2, \ldots, 2024\}$. What is the probability that j is divisible by 9 and 15?

(A) $\frac{1}{23}$ (B) $\frac{1}{46}$

(C) $\frac{1}{44}$ (D) $\frac{1}{253}$

$\#\, n(S) = 2024$

$n(A) = 44$

$P(A) = \frac{44}{2024} = \frac{11}{506} = \frac{1}{46}$

45

$\frac{2024}{45} = 44.9$

3. Let S_n be the set of all n-digit numbers whose digits are all 1 or 2 and there are no consecutive 2's. (Example: 112 is in S_3 but 221 is not in S_3). Then the number of elements in S_{10} is

(A) 512 (B) 256 (C) 144 (D) 89

$S_n = S_{n-1} + S_{n-2}$

$S_1 = \{1, 2\} = 2$

$S_2 = \{11, 12, 21\} = 3$



2, 3, 5, 8, 13, 21, 34, 55, 89, S_{10} 144

4. There are 30 True or False questions in an examination. A student knows the answer to 20 questions and guesses the answers to the remaining 10 questions at random. What is the probability that the student gets exactly 24 answers correct?

(A) $\frac{105}{2^9}$ (B) $\frac{105}{2^8}$ (C) $\frac{105}{2^{10}}$ (D) $\frac{4}{2^{10}}$

10 problem $p=\frac{1}{2}$ $q=\frac{1}{2}$

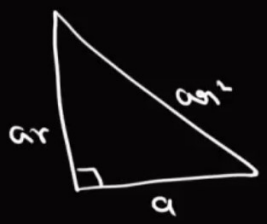
$P(X=4) = {}^{10}C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^6$

$= {}^{10}C_4 \left(\frac{1}{2}\right)^{10}$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{1}{2^{10}} = \frac{105}{2^9}$

5. Let T be a right-angled triangle in the plane whose side lengths are in a geometric progression. Let $n(T)$ denote the number of sides of T that have integer lengths. Then the maximum value of $n(T)$ over all such T is

(A) 0 (B) 1 (C) 2 (D) 3



$a^2 + (ar)^2 = (ar^2)^2$

$r^4 - r^2 - 1 = 0$

$r^2 = \frac{1 \pm \sqrt{5}}{2}$ $r = \sqrt{\frac{1+\sqrt{5}}{2}}$

$r \neq 0$

$r^2 \neq 0$

(i) $a \in \mathbb{Z}$ $ar, ar^2 \notin 0$

(ii) $ar \in \mathbb{Z} = k$ $\frac{k}{r}, kr \notin 0$

(iii) $ar^2 \in \mathbb{Z} = k$ $\frac{k}{r^2}, \frac{k}{r^4} \notin 0$

6. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that $\sum_{i=1}^{n} x_i = 1$. What is the maximum possible value of $\sum_{i=1}^{n} \sqrt{x_i}$?

(A) 1 (B) $\sqrt{n}$ (C) $n^{3/4}$ (D) n

$x_1 + x_2 + \cdots + x_n = 1$ $\sum_{i=1}^{n} \sqrt{x_i}$

$(1 \cdot \sqrt{x_1} + 1 \cdot \sqrt{x_2} + \cdots + 1 \cdot \sqrt{x_n})^2 \leq (x_1 + x_2 + \cdots + x_n)(1 + 1 + \cdots + 1) = n$

$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq \sqrt{n}$

7. The precise interval on which the function $f(x) = \log_{1/2}(x^2 - 2x - 3)$ is monotonically decreasing, is

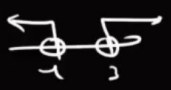
(A) $(-\infty, -1)$ (B) $(-\infty, 1)$

(C) $(1, \infty)$ (D) $(3, \infty)$

Domain:

$x^2 - 2x - 3 > 0$

$(x-3)(x+1) > 0$



$f(x) = \dfrac{\ln(x^2-2x-3)}{\ln(0.5)}$

$f'(x) = \dfrac{1}{\ln(0.5)} \cdot \dfrac{2(x-1)}{x^2-2x-3} < 0$

$\ln(0.5) < 0$



8. The angle subtended at the origin by the common chord of the circles $x^2 + y^2 - 6x - 6y = 0$ and $x^2 + y^2 = 36$ is

(A) $\pi/2$ (B) $\pi/4$ (C) $\pi/3$ (D) $2\pi/3$

$(3,3)$



$S - S' = 0$

$x + y = 6$

$-6x - 6y + 36 = 0$

$x + y = 6$

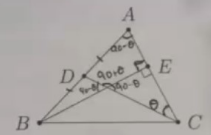


$x^2 + y^2 = 2\left(\dfrac{x+y}{2}\right)^2$

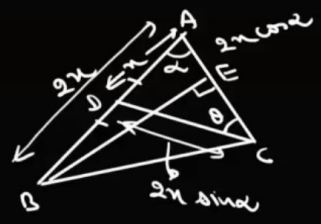
$xy = 0$

$\theta = \dfrac{\pi}{2}$

9. In $\triangle ABC$, CD is the median and BE is the altitude. Given that $\overline{CD} = \overline{BE}$, what is the value of $\angle ACD$?



(A) $\pi/3$ (B) $\pi/4$ (C) $\pi/5$ (D) $\pi/6$

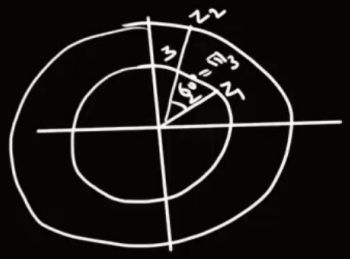


$AE = 2x\cos\alpha \quad BE = CD = 2x\sin\alpha$

$\dfrac{\sin\alpha}{2x\sin\alpha} = \dfrac{\sin\theta}{x} \quad \sin\theta = \dfrac{1}{2} \quad \theta = \dfrac{\pi}{6}$

10. If the points z_1 and z_2 are on the circles $|z| = 2$ and $|z| = 3$, respectively, and the angle included between these vectors is $60°$, then the value of $\frac{|z_1+z_2|}{|z_1-z_2|}$ is

(A) $\sqrt{\frac{19}{7}}$ (B) $\sqrt{19}$ (C) $\sqrt{7}$ (D) $\sqrt{\frac{7}{19}}$



$z_1 = 2e^{i\theta}$

$z_2 = 3e^{i(\theta+\frac{\pi}{3})}$

$$\left|\frac{z_1+z_2}{z_1-z_2}\right| = \left|\frac{2e^{i\theta}+3e^{i(\theta+\frac{\pi}{3})}}{2e^{i\theta}-3e^{i(\theta+\frac{\pi}{3})}}\right|$$

$$= \left|\frac{2+3e^{i\frac{\pi}{3}}}{2-3e^{i\frac{\pi}{3}}}\right| = \left|\frac{2+\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{2-\frac{3}{2}-\frac{3\sqrt{3}}{2}i}\right|$$

$$= \left|\frac{7+3\sqrt{3}i}{1-3\sqrt{3}i}\right| = \sqrt{\frac{76}{28}} = \sqrt{\frac{19}{7}}$$

11. Let $n \geqslant 1$. The maximum possible number of primes in the set $\{n+6, n+7, \ldots, n+34, n+35\}$ is

(A) 7. (B) 8 (C) 12 (D) 13

$6n+1$, $6n+5$

$a=1$ $a=6$

$\textcircled{5} + \textcircled{5} = \textcircled{10}$

no com.

$\{n+6, n+7, \ldots n+35\}$

30 numbers $\textcircled{7}$

$n \geq 1$

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

2 4 2 4 2 4 2 6

$n=10$. #8.

12. Suppose 40 distinguishable balls are to be distributed into 4 different boxes such that each box gets exactly 10 balls. Out of these 40 balls, 10 are defective and 30 are non-defective. In how many ways can the balls be distributed such that all the defective balls go to the first two boxes?

(A) $\frac{40!}{(10!)^4}$ (B) $\frac{30! \cdot 20!}{(10!)^5}$ (C) $\frac{20! \cdot 20!}{(10!)^5}$ (D) $\frac{30! \cdot 10!}{(10!)^4}$



$$= \frac{30!}{10!\,20!} \cdot \frac{20!}{10!\,10!} \cdot \frac{20!}{10!\,10!}$$

$$= \frac{30!\,20!}{(10!)^5}$$

$$\frac{30!}{(10!)^3\,2!}$$

13. The number of elements in the set

$$\{x : 0 \leqslant x \leqslant 2, |x - x^5| = |x^5 - x^6|\}$$

is

(A) 2 (B) 3 (C) 4 (D) 5

$30C_{10}$

$|x - x^5| = |x^5 - x^6|$

$0 \le x \le 2$

$x - x^5 = x^5 - x^6$

$x^6 - 2x^5 + x = 0$

$x \in [0,1]$

$x - x^5 \geq 0$

$x^5 - x^6 \geq 0$

$x \in [1,2)$ $x - x^5 \leq 0$

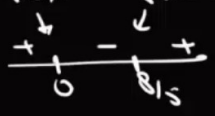
$x^5 - x^6 \leq 0$

$\boxed{x=0}$ $x^5 - 2x^4 + 1 = 0$.

let $f(x) = x^5 - 2x^4 + 1$

$f'(x) = 5x^4 - 8x^3$

$= x^3(5x - 8)$



$x \in [0,1]$

$x - x^5 \geq 0$

$x^5 - x^6 \geq 0$

$x \in [1,2)$ $x - x^5 \leq 0$

$x^5 - x^6 \leq 0$

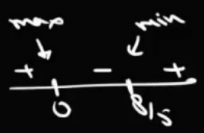
$x^6 - 2x^5 + x = 0$

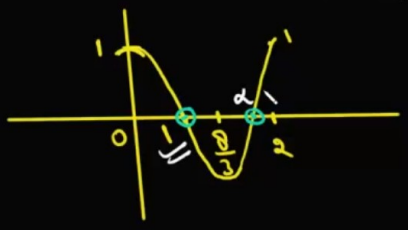
$\boxed{x=0}$ $x^5 - 2x^4 + 1 = 0$.

let $f(x) = x^5 - 2x^4 + 1$

$f'(x) = 5x^4 - 8x^3$

$= x^3(5x - 8)$





$f(0) = 1$ $f(2) = 1$

$f(1) = 0$





13. The number of elements in the set

$$\{x : 0 \leqslant x \leqslant 2, |x - x^5| = |x^5 - x^6|\}$$

is

(A) 2 (B) 3 (C) 4 (D) 5

$30C_{10}$

$|x - x^5| = |x^5 - x^6|$

$0 \le x \le 2$

$x - x^5 = x^5 - x^6$

14. In a room with $n \geq 2$ people, each pair shakes hands between themselves with probability $\frac{2}{n^2}$ and independently of other pairs. If p_n is the probability that the total number of handshakes is at most 1, then $\lim_{n\to\infty} p_n$ is equal to

(A) 0 (B) 1 (C) e^{-1} (D) $2e^{-1}$

n

No. of handshakes $= {}^nC_2 = N = \frac{n(n-1)}{2}$

$p = \frac{2}{n^2}$ $\quad q = 1 - \frac{2}{n^2}$

$P(X \leq 1) = {}^NC_0 q^N + {}^NC_1 q^{N-1} p$

$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \left(1 - \frac{2}{n^2}\right)^{\frac{n^2}{2}\cdot\frac{2}{n^2}\cdot\frac{n^2}{2}} + \frac{n^2}{2}\left(1 - \frac{2}{n^2}\right)^{\frac{n^2}{2}-2}\cdot\frac{2}{n^2}$

$= e^{\lim_{n\to\infty} -\frac{2}{n^2}\cdot\frac{n^2}{2}\cdot\frac{2}{n^2}} + \lim_{n\to\infty}\frac{n^2}{n^2}\, e^{\lim_{n\to\infty} -\frac{2}{n^2}\cdot\frac{n^2}{2}\cdot\frac{2}{n^2}}$

$= e^{-1} + 1 \cdot e^{-1} = 2e^{-1}$

15. The number of positive solutions to the equation

$$e^x \sin x = \log x + e^{\sqrt{x}} + 2$$

is

(A) 0 (B) 1 (C) 2 (D) ∞

$e^x \sin x = \underbrace{\ln x + e^{\sqrt{x}} + 2}_{g(x)}$



$y' = \frac{1}{x} + \frac{e^{\sqrt{x}}}{2\sqrt{x}} > 0$

* $e^x = \ln x + e^{\sqrt{x}} + 2$ # No. of soln

$e^x - e^{\sqrt{x}} = \ln x + 2$

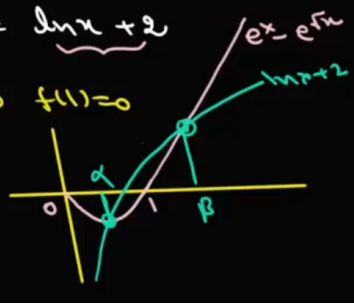
two real roots

$f(x) = e^x - e^{\sqrt{x}}$ $\quad f(0) = 0$ $\quad f(1) = 0$

$x \in [0,1]$ $\quad e^x \leq e^{\sqrt{x}}$

$e^x - e^{\sqrt{x}} \leq 0$

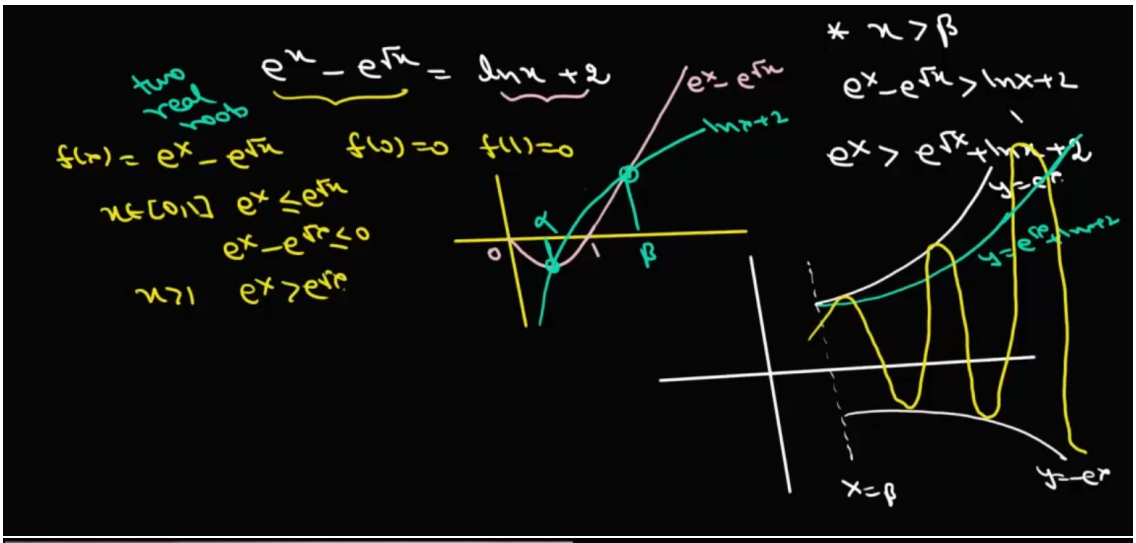
$x > 1$ $\quad e^x > e^{\sqrt{x}}$



* $x > \beta$

$e^x - e^{\sqrt{x}} > \ln x + 2$

$e^x > e^{\sqrt{x}} + \ln x + 2$

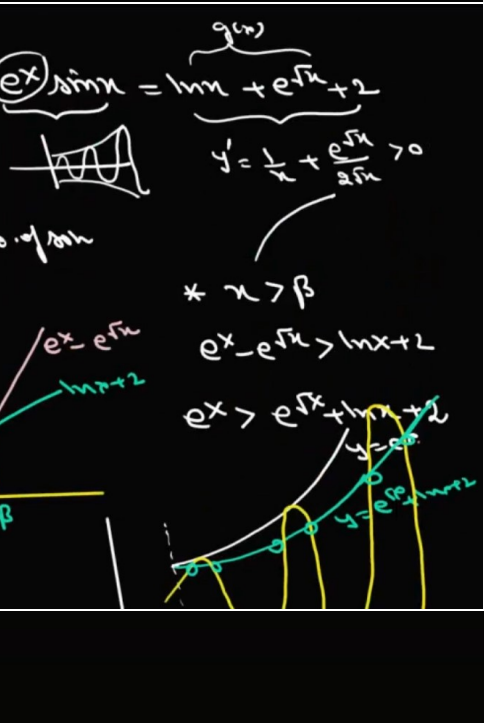


15. The number of positive solutions to the equation

$$e^x \sin x = \log x + e^{\sqrt{x}} + 2$$

is

(A) 0 (B) 1 (C) 2 (D) ∞



16. Let $n > 1$ be the smallest composite integer that is coprime to $\frac{10000!}{9900!}$. Then

(A) $n \leqslant 100$ (B) $100 < n \leqslant 9900$

(C) $9900 < n \leqslant 10000$ (D) $n > 10000$

$$\frac{10000!}{9900!} = \underbrace{(9901)(9902)\cdots(10000)}_{\text{product of 100 numbers}}$$

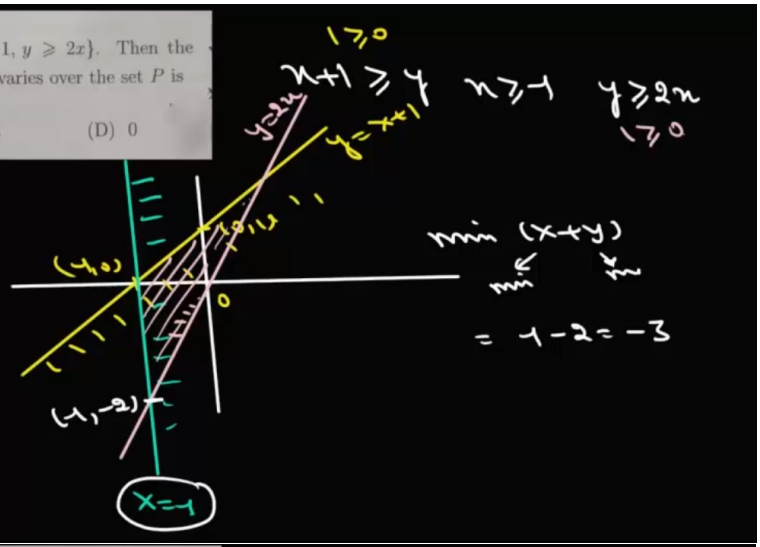
100!

$N = p \times q$

$N > 10000$ $p > 100$ $q > 100$

17. Let $P = \{(x, y) : x + 1 \geqslant y, x \geqslant -1, y \geqslant 2x\}$. Then the minimum value of $(x + y)$ where (x, y) varies over the set P is

(A) -1 (B) -3 (C) 3 (D) 0

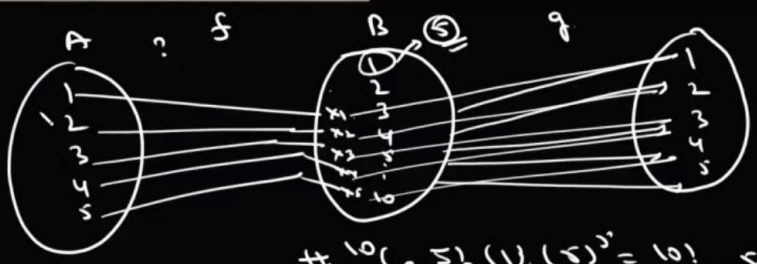


18. Let $A = \{1, \ldots, 5\}$ and $B = \{1, \ldots, 10\}$. Then the number of ordered pairs (f, g) of functions $f : A \to B$ and $g : B \to A$ satisfying $(g \circ f)(a) = a$ for all $a \in A$ is

(A) $\frac{10!}{5!} \times 5^5$ (B) $5^{10} \times 5!$ (C) $10! \times 5!$ (D) $\binom{10}{5} \times 10^5$

$gof(a) = a$

$g(f(a)) = a$



$\#^{10}C_5 \cdot 5! \cdot (1) \cdot (5)^5 = \frac{10!}{5!5!} \cdot 5! \cdot 5^5$

$= \frac{10!}{5!}(5^5)$

19. Let

$$S = \frac{1}{\sqrt{10000}} + \frac{1}{\sqrt{10001}} + \cdots + \frac{1}{\sqrt{160000}}.$$

Then the largest positive integer not exceeding S is

(A) 200 (B) 400 (C) 600 (D) 800

$$S = \frac{1}{\sqrt{10000}} + \frac{1}{\sqrt{10001}} + \cdots + \frac{1}{\sqrt{160000}}$$

$$\sqrt{n} + \sqrt{n-1} < 2\sqrt{n} < \sqrt{n} + \sqrt{n+1}$$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n-1}}$$

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}$$

$$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1})$$

$$\sqrt{n+1}-\sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n}-\sqrt{n-1}$$

$$\sum_{n=10000}^{160000} 2\left(\sqrt{n+1}-\sqrt{n}\right) < \underbrace{\sum_{n=10000}^{160000} \frac{1}{\sqrt{n}}}_{S} < \sum_{n=10000}^{160000} 2\left(\sqrt{n}-\sqrt{n-1}\right)$$

$$2\left(\sqrt{160001}-100\right) < S < 2\left(400-\sqrt{9999}\right)$$

$$600.003 < S < 600.01$$

19. Let

$$S = \frac{1}{\sqrt{10000}} + \frac{1}{\sqrt{10001}} + \cdots + \frac{1}{\sqrt{160000}}.$$

Then the largest positive integer not exceeding S is

(A) 200 (B) 400 (C) 600 (D) 800

20. The real number x satisfies

$$\frac{|x|^2-|x|-2}{2|x|-|x|^2-2} > 2$$

if and only if x belongs to

(A) $(-2,-1)\cup(1,2)$ (B) $(-2/3,0)\cup(0,2/3)$

(C) $(-1,-2/3)\cup(2/3,1)$ (D) $(-1,0)\cup(0,1)$

$$-\frac{|x|^2-|x|-2}{|x|^2-2|x|+2} > 2 \qquad (|x|-1)^2+1$$

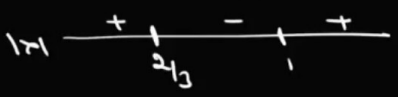
$$-|x|^2+|x|+2 > 2|x|^2-4|x|+4$$

$$3|x|^2-5|x|+2 < 0$$

$$(3|x|-2)(|x|-1) < 0$$

$$\frac{2}{3} < |x| < 1$$

$$\frac{2}{3} < x < 1 \quad \text{or} \quad -1 < x < -\frac{2}{3}$$



21. Consider points of the form (n, n^k), where n and k are integers with $n \geq 0, k \geq 1$. How many such points are strictly inside the circle of radius 10 with centre at the origin?

(A) 11 (B) 12 (C) 15 (D) 17

(n, n^k) $n \geq 0$ $k \geq 1$



$x^2 + y^2 < 10^2$

$n^2 + n^{2k} < 100$ $\quad n^{2k} < 100 - n^2$

$n=0$ $(0,0)$

$n=1$ $(1,1)$

$n=2$ $(2,2)$ $(2,4)$, $(2,8)$ $\quad 16^k < 86$

$n=3$ $(3,3)$ $(3,9)$ $\quad 9^k < 91$, $k < 2$

$n=4$

$2^{2k} < 96$

$4^k < 96$ $\quad k \leq 3$

$n^2 + n^{2k} < 100$ $\quad n^{2k} < 100 - n^2$

$n=0$ $(0,0)$

$n=1$ $(1,1)$

$n=2$ $(2,2)$ $(2,4)$, $(2,8)$ $\quad 16^k < 86$

$n=3$ $(3,3)$ $(3,9)$ $\quad 9^k < 91$, $k < 2$

$n=4$ $(4,4)$ $\quad 25^k < 78$

$n=5$ $(5,5)$ $\quad 36^k < 64$

$n=6$ $(6,6)$ $\quad 49^k < 51$

$n=7$ $(7,1)$

$2^{2k} < 96$

$4^k < 96$ $\quad k \leq 3$

21. Consider points of the form (n, n^k), where n and k are integers with $n \geq 0, k \geq 1$. How many such points are strictly inside the circle of radius 10 with centre at the origin?

(A) 11 (B) 12 (C) 15 (D) 17

22. Let $n > 1$, and let us arrange the expansion of $\left(x^{1/2} + \frac{1}{2x^{1/4}}\right)^n$ in decreasing powers of x. Suppose the first three coefficients are in arithmetic progression. Then, the number of terms where x appears with an integer power, is

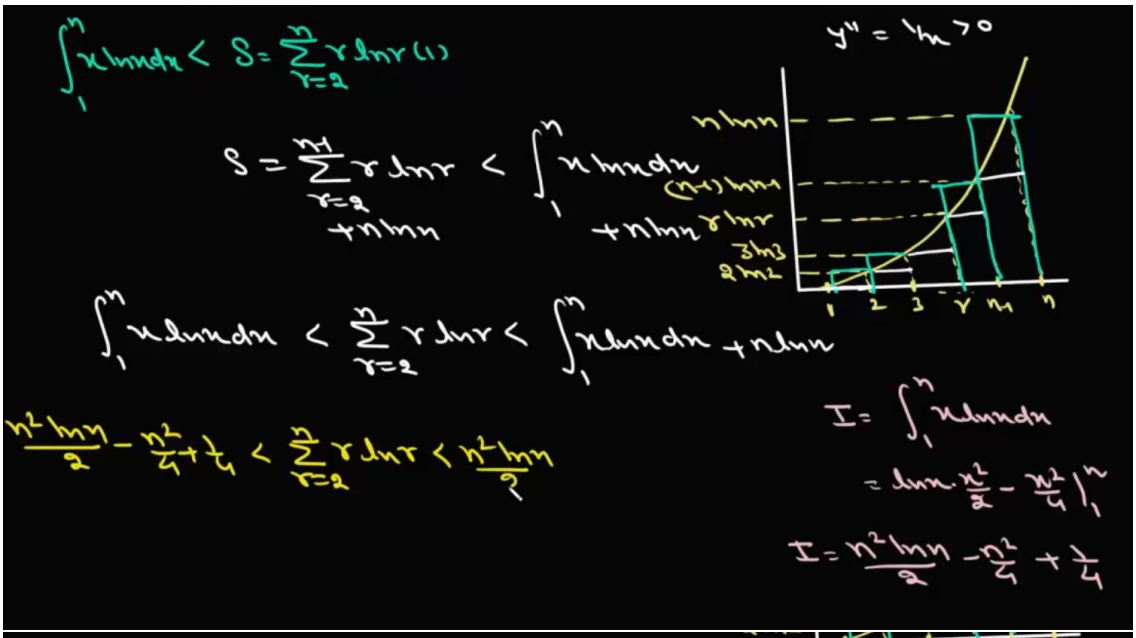
(A) 3 (B) 2 (C) 1 (D) 0

23. The limit

$$\lim_{n\to\infty} \frac{2\log 2 + 3\log 3 + \cdots + n\log n}{n^2 \log n}$$

is equal to

(A) 0 (B) 1/4 (C) 1/2 (D) 1



$$\int_1^n x\ln x\,dx < S = \sum_{r=2}^{n} r\ln r \quad (1)$$

$$S = \sum_{r=2}^{n-1} r\ln r + n\ln n < \int_1^n x\ln x\,dx + n\ln n$$

$$\int_1^n x\ln x\,dx < \sum_{r=2}^{n} r\ln r < \int_1^n x\ln x\,dx + n\ln n$$

$$\frac{n^2}{2}\ln n - \frac{n^2}{4} + \frac{1}{4} < \sum_{r=2}^{n} r\ln r < \frac{n^2}{2}\ln n$$

$$I = \int_1^n x\ln x\,dx = \ln x\cdot\frac{x^2}{2}\Big|_1^n - \frac{x^2}{4}\Big|_1^n$$

$$I = \frac{n^2}{2}\ln n - \frac{n^2}{4} + \frac{1}{4}$$

$$\int_1^n x\ln x\,dx < \sum_{r=2}^{n} r\ln r < \int_1^n x\ln x\,dx + n\ln n$$

$$\frac{\frac{n^2}{2}\ln n - \frac{n^2}{4} + \frac{1}{4}}{\frac{n^2}{2}\ln n} < \frac{\sum_{r=2}^{n} r\ln r}{\frac{n^2}{2}\ln n} < \frac{\frac{n^2}{2}\ln n - \frac{n^2}{4} + \frac{1}{4} + n\ln n}{\frac{n^2}{2}\ln n}$$

$$\lim_{n\to\infty}\left(1 - \frac{1}{2\ln n} + \frac{1}{2n^2\ln n}\right) < \lim_{n\to\infty}\frac{\sum_{r=2}^{n} r\ln r}{\frac{n^2}{2}\ln n} < \lim_{n\to\infty}\left(1 - \frac{1}{2\ln n} + \frac{1}{2n^2\ln n} + \frac{2}{n}\right)$$

$$1 < \lim_{n\to\infty}\frac{\sum_{r=2}^{n} r\ln r}{\frac{n^2}{2}\ln n} < 1$$

$$\therefore \lim_{n\to\infty}\frac{\sum_{r=2}^{n} r\ln r}{\frac{n^2}{2}\ln n} = 1$$

23. The limit

$$\lim_{n\to\infty}\frac{2\log 2 + 3\log 3 + \cdots + n\log n}{n^2\log n}$$

is equal to

(A) 0 (B) 1/4 (C) 1/2 (D) 1

24. Let $p < q$ be prime numbers such that $p^2 + q^2 + 7pq$ is a perfect square. Then, the largest possible value of q is:

(A) 7 (B) 11 (C) 23 (D) 29

$= x^2$

$(p+q)^2 + 5pq$

$5pq = x^2 - (p+q)^2$

$p^2 + q^2 + 7pq = n^2$

$(p+q)^2 + 5pq = n^2$

$(n + p+q)(n - (p+q)) = 5pq.$

$n + p+q = 5^a p^b q^c$

$n - (p+q) = 5^{1-a} \cdot p^{1-b} \cdot q^{1-c}$

$5pq = x^2 - (p+q)$

$p^2 + q^2 + 7pq = n^2$

$(p+q)^2 + 5pq = n^2$

$(n + p+q)(n - (p+q)) = 5pq.$

$n + p+q = 5^a p^b q^c$

$n - (p+q) = 5^{1-a} \cdot p^{1-b} \cdot q^{1-c}$

$2(p+q) = 5^a \cdot p^b q^c - 5^{1-a} \cdot p^{1-b} \cdot q^{1-c}.$

$pq - 5$

$5pq$

$5pq + 1$

$1 - 5pq$

① $2p + 2q = pq - 5$

$2p + 2q - pq = -5$

$p(2-q) - 2(2-q) = -9$

$(p-2)(q-2) = 9$

$p-2 = 1 \quad q-2 = 9$

$p = 3, \quad q = 11$

$a, b, c \in \{0, 1\}$

②

24. Let $p < q$ be prime numbers such that $p^2 + q^2 + 7pq$ is a perfect square. Then, the largest possible value of q is:

(A) 7 (B) 11 (C) 23 (D) 29

25. The set of all real numbers x for which $3^{2^{1-x^2}}$ is an integer has

(A) 3 elements (B) 15 elements

(C) 24 elements (D) infinitely many elements

26. Let a, b, c be three complex numbers. The equation

$$az + b\bar{z} + c = 0$$

represents a straight line on the complex plane if and only if

(A) $a = b$ (B) $\bar{a}c = b\bar{c}$

(C) $|a| = |b| \neq 0$ (D) $|a| = |b| \neq 0$ and $\bar{a}c = b\bar{c}$

$az + b\bar{z} + c = 0$

$\left(\frac{a}{c}\right)z + \left(\frac{b}{c}\right)\bar{z} + 1 = 0$

$\frac{a}{c} = \left(\frac{\bar{b}}{\bar{c}}\right)$

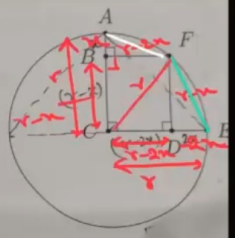
$a\bar{c} = c\bar{b}$

$\bar{a}c = \bar{c}b$

$a\bar{z} + \bar{a}z + b = 0$

$a + c \; b \in \mathbb{R}$

27. In the adjoining figure, C is the centre of the circle drawn, A, F, E lie on the circle and $BCDF$ is a rectangle. If $\frac{DE}{AB} = 2$, then $\frac{FE}{FA}$ equals



(A) $\sqrt{\frac{3}{2}}$ (B) $\sqrt{2}$

(C) $\sqrt{\frac{5}{2}}$ (D) $\sqrt{3}$

$4r^2 =$

$(r-x)^2 + (r-2x)^2 = r^2 \quad -(i)$

$FE = \sqrt{(r-x)^2 + (2x)^2}$

$= \sqrt{r^2 - (r-2x)^2 + (2x)^2}$

$= \sqrt{4xr} = 2\sqrt{xr}$

$FA = \sqrt{x^2 + (r-2x)^2} = \sqrt{x^2 + r^2 - (r-x)^2} = \sqrt{2rx}$

$\frac{FE}{FA} = \frac{2\sqrt{xr}}{\sqrt{2rx}} = \sqrt{2}$

28. For every increasing function $b : [1, \infty) \to [1, \infty)$ such that

$$\int_1^\infty \frac{dx}{b(x)} < \infty,$$

we must have

(A) $\sum_{k=1}^{\infty} \frac{\sqrt{\log k}}{b(k)} < \infty$

(B) $\sum_{k=3}^{\infty} \frac{\log k}{b(\log k)} < \infty$

(C) $\sum_{k=1}^{\infty} \frac{e^k}{b(e^k)} < \infty$

(D) $\sum_{k=3}^{\infty} \frac{1}{\sqrt{b(\log k)}} < \infty$

$\int_1^\infty \frac{dx}{b(x)} < \infty$

Let $x = e^t$

$dx = e^t dt$

$\int_0^\infty \frac{e^t dt}{b(e^t)} < \infty$

$\int_1^\infty \frac{e^t dt}{b(e^t)} < \infty$

$\sum_{k=1}^{\infty} \frac{e^k}{b(e^k)} < \infty$

29. Consider the following two statements:

(I) There exists a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x^3 + x^5) = e^x - 100$.

(II) There exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(e^x) = x^3 + x^5$.

Then

(A) Only (I) is correct.

(B) Only (II) is correct.

(C) Both (I) and (II) are correct.

(D) Neither (I) nor (II) is correct.

$g'(x^3+x^5)(3x^2+5x^4) = e^x$

$g(x^3+x^5) = e^x - 100$

$g'(x^3+x^5) = \frac{e^x}{3x^2+5x^4}$

$g'(0)$?

$g(e^x) = x^3 + x^5$ $\quad e^x = t$

$g(t) = (\ln t)^3 + (\ln t)^5$ $\quad t > 0$

$\lim_{x \to 0^+} g(x) = -\infty$

30. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\int_{-1}^{1} \psi(x)dx = 1$.
Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then

$$\lim_{x \to 0} \frac{1}{x} \int_{1-x}^{1+x} f(y)\psi\left(\frac{1-y}{x}\right) dy$$

equals

(A) $f(1)$

(B) $f(1)\psi(0)$

(C) $f'(1)\psi(0)$

(D) $f(1)\psi(1)$

$$\int_{-1}^{1} \psi(x)\,dx = 1$$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{1-\epsilon}^{1+\epsilon} f(y)\,\psi\left(\frac{1-y}{\epsilon}\right) dy$$

Let $\frac{1-y}{\epsilon} = t \qquad y = 1 - \epsilon t$

$$\frac{dy}{\epsilon} = -dt$$

$$\lim_{\epsilon \to 0} -\int_{1}^{-1} f(y)\,\psi(t)\,dt$$

$$= \lim_{\epsilon \to 0} \int_{-1}^{1} f(1-\epsilon t)\,\psi(t)\,dt = \int_{-1}^{1} f(1)\,\psi(t)\,dt = f(1)\int_{-1}^{1} \psi(t)\,dt$$

$$= f(1)$$