1. The number of isosceles (but not equilateral) triangles with integer sides and no side exceeding 10 is

(A) 65 (B) 75 (C) 81 (D) 90.

The number of solutions (x, y, z) to the equation $x - y - z = 25$, where x, y, and z are positive integers such that $x \le 40$, $y \le 12$, and $z \le 12$ is

A. 101

B. 99

C. 87

D. 105

5. An alien script has n letters $b_1, \ldots, b_n$. For some $k < n/2$ assume that all words formed by any of the k letters (written left to right) are meaningful. These words are called *k-words*. A k-word is considered **sacred** if:

   i) no letter appears twice and,

   ii) if a letter b_i appears in the word then the letters b_{i-1} and b_{i+1} do not appear. (Here $b_{n+1} = b_1$ and $b_0 = b_n$.)

   For example, if $n = 7$ and $k = 3$ then $b_1b_3b_6, b_3b_1b_6, b_2b_4b_6$ are sacred 3-words. On the other hand $b_1b_7b_4, b_2b_2b_6$ are not sacred. What is the total number of sacred k-words? Use your formula to find the answer for $n = 10$ and $k = 4$.

7. Let N be a positive integer such that $N(N - 101)$ is the square of a positive integer. Then determine all possible values of N. (Note that 101 is a prime number).

1 A class has 100 students. Let a_i, $1 \le i \le 100$, denote the number of friends the i-th student has in the class. For each $0 \le j \le 99$, let c_j denote the number of students having *at least* j friends. Show that

$$\sum_{i=1}^{100} a_i = \sum_{j=1}^{99} c_j .$$

11. Let $S(k)$ denote the set of all one-to-one and onto functions from $\{1, 2, 3, \ldots, k\}$ to itself. Let p, q be positive integers. Let $S(p, q)$ be the set of all τ in $S(p+q)$ such that $\tau(1) < \tau(2) < \cdots < \tau(p)$ and $\tau(p+1) < \tau(p+2) < \cdots < \tau(p+q)$. The number of elements in the set $S(13, 29)$ is

(A) 377. (B) (42)!. (C) $\binom{42}{13}$. (D) $\frac{42!}{29!}$.

12. Suppose that both the roots of the equation $x^2 + ax + 2016 = 0$ are positive even integers. The number of possible values of a is

(A) 6. (B) 12. (C) 18. (D) 24.

---

22. The number of 3-digit numbers abc such that we can construct an isosceles triangle with sides a, b and c is

(A) 153. (B) 163. (C) 165. (D) 183.

8 Consider $n\,(>1)$ lotus leaves placed around a circle. A frog jumps from one leaf to another in the following manner. It starts from some selected leaf. From there, it skips exactly one leaf in the clockwise direction and jumps to the next one. Then it skips exactly two leaves in the clockwise direction and jumps to the next one. Then it skips three leaves again in the clockwise direction and jumps to the next one, and so on. Notice that the frog may visit the same leaf more than once. Suppose it turns out that if the frog continues this way, then all the leaves are visited by the frog sometime or the other. Show that n *cannot* be odd.

9. Let $A = \{x_1, x_2, \ldots, x_{50}\}$ and $B = \{y_1, y_2, \cdots, y_{20}\}$ be two sets of real numbers. What is the total number of functions $f: A \to B$ such that f is onto and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{50})$?

(A) $\binom{49}{19}$ (B) $\binom{49}{20}$ (C) $\binom{50}{19}$ (D) $\binom{50}{20}$

14. Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain any two consecutive numbers. Then

(A) $a_n = a_{n-1} + a_{n-2}$

(B) $a_n = 2a_{n-1}$

(C) $a_n = a_{n-1} - a_{n-2}$

(D) $a_n = a_{n-1} + 2a_{n-2}$.

7. Consider a right-angled triangle with integer-valued sides $a < b < c$ where a, b, c are pairwise co-prime. Let $d = c - b$. Suppose d divides a. Then

(a) Prove that $d \leq 2$.

(b) Find all such triangles (i.e. all possible triplets a, b, c) with perimeter less than 100.

1. The number of ways one can express $2^2 3^3 5^5 7^7$ as a product of two numbers a and b, where $gcd(a, b) = 1$, and $1 < a < b$, is

(A) 5. (B) 6. (C) 7. (D) 8.

14. The number of different ways to colour the vertices of a square $PQRS$ using one or more colours from the set {Red, Blue, Green, Yellow} such that no two adjacent vertices have the same colour is

(A) 36. (B) 48. (C) 72. (D) 84.

13. Define $a = p^3 + p^2 + p + 11$ and $b = p^2 + 1$, where p is any prime number. Let $d = gcd(a, b)$. Then the set of possible values of d is

(A) $\{1, 2, 5\}$. (B) $\{2, 5, 10\}$. (C) $\{1, 5, 10\}$. (D) $\{1, 2, 10\}$.

15. Let a, b, c and d be four non-negative real numbers where $a+b+c+d = 1$. The number of different ways one can choose these numbers such that $a^2 + b^2 + c^2 + d^2 = \max\{a, b, c, d\}$ is

(A) 1. (B) 5. (C) 11. (D) 15

8. A finite sequence of numbers $(a_1, \ldots, a_n)$ is said to be *alternating* if

$$a_1 > a_2, \quad a_2 < a_3, \quad a_3 > a_4, \quad a_4 < a_5, \ldots$$

$$\text{or} \quad a_1 < a_2, \quad a_2 > a_3, \quad a_3 < a_4, \quad a_4 > a_5, \ldots$$

How many alternating sequences of length 5, with *distinct* numbers $a_1, \ldots, a_5$ can be formed such that $a_i \in \{1, 2, \ldots, 20\}$ for $i = 1, \ldots, 5$?

4. The number of factors of $2^{15} \times 3^{10} \times 5^6$ which are either perfect squares or perfect cubes (or both) is:

(A) 264 (B) 260 (C) 256 (D) 252

23 Consider all the permutations of the twenty six English letters that start with z. In how many of these permutations the number of letters between z and y is less than those between y and x?

(A) $6 \times 23!$ (B) $6 \times 24!$ (C) $156 \times 23!$ (D) $156 \times 24!$.

11. When the product of four consecutive odd positive integers is divided by 5, the set of remainder(s) is

(A) $\{0\}$ (B) $\{0, 4\}$ (C) $\{0, 2, 4\}$ (D) $\{0, 2, 3, 4\}$

Question 54

How many different numbers can be formed by using only the digits 1 and 3 which are smaller than 3000000?

a) 64
b) 128
c) 190
d) 254

12. Consider the equation $x^2 + y^2 = 2015$ where $x \geq 0$ and $y \geq 0$. How many solutions (x, y) exist such that both x and y are non-negative integers?

(A) Greater than two
(B) Exactly two
(C) Exactly one
(D) None

Question 55

There are n numbers $a_1, a_2, a_3, \ldots, a_n$ each of them being +1 or -1. If it is known that $a_1a_2 + a_2a_3 + a_3a_4 + .. a_{n-1}a_n + a_na_1 = 0$ then

a) n is a multiple of 2 but not a multiple of 4
b) n is a multiple of 3
c) n can be any multiple of 4
d) The only possible value of n is 4

6. Consider all functions $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ which are one-one, onto and satisfy the following property:

$$\text{if } f(k) \text{ is odd then } f(k+1) \text{ is even, } k = 1, 2, 3.$$

The number of such functions is
(A) 4 (B) 8 (C) 12 (D) 16.

20. Let $ABCDEFGHIJ$ be a 10-digit number, where all the digits are distinct. Further, $A > B > C$, $A + B + C = 9$, $D > E > F > G$ are consecutive odd numbers and $H > I > J$ are consecutive even numbers. Then A is

(A) 5 (B) 6 (C) 7 (D) 8

The cancellation of the Wimbledon tournament has led to a world surplus of tennis balls, and Santa has decided to use them as stocking fillers. He comes down the chimney with n identical tennis balls, and he finds k named stockings waiting for him.

Let $g(n,k)$ be the number of ways that Santa can put the n balls into the k stockings; for example, $g(2,2) = 3$, because with two balls and two children, Miriam and Adam, he can give both balls to Miriam, or both to Adam, or he can give them one ball each.

(i) What is the value of $g(1,k)$ for $k \geqslant 1$?

(ii) What is the value of $g(n,1)$?

(iii) If there are $n \geqslant 2$ balls and $k \geqslant 2$ children, then Santa can either give the first ball to the first child, then distribute the remaining balls among all k children, or he can give the first child none, and distribute all the balls among the remaining children. Use this observation to formulate an equation relating the value of $g(n,k)$ to other values taken by g.

(iv) What is the value of $g(7,5)$?

(v) After the first house, Rudolf reminds Santa that he ought to give at least one ball to each child. Let $h(n,k)$ be the number of ways of distributing the balls according to this restriction. What is the value of $h(7,5)$?

21. Let $A = \{(a,b,c) : a, b, c \text{ are prime numbers}, a < b < c, a+b+c = 30\}$. The number of elements in A is

(A) 3 (B) 2 (C) 1 (D) 0

10. Let S be a set of n elements. The number of ways in which n distinct non-empty subsets $X_1, ..., X_n$ of S can be chosen such that $X_1 \subseteq X_2 \cdots \subseteq X_n$, is

(A) $\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}$ (B) 1 (C) $n!$ (D) 2^n

12. The number of distinct even divisors of

$$\prod_{k=1}^{5} k!$$

is

(A) 24 (B) 32 (C) 64 (D) 72

25. Let

$$A = \begin{bmatrix} a & 1 & 1 \\ b & a & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The number of elements in the set

$$\left\{(a,b) \in \mathbb{Z}^2 : 0 \le a, b \le 2021,\ \text{rank}(A) = 2\right\}$$

is

(A) 2021 (B) 2020 (C) $2021^2 - 1$ (D) 2020×2021

This question is about counting the number of ways of partitioning a set of n elements into subsets, each with at least two and at most n elements. If n and k are integers with $1 \leqslant k \leqslant n$, let $f(n,k)$ be the number of ways of partitioning a set of n elements into k such subsets. For example, $f(5,2) = 10$ because the allowable partitions of $\{1,2,3,4,5\}$ are

$$\begin{array}{ll} \{1,5\},\{2,3,4\}, & \{1,2,5\},\{3,4\}, \\ \{2,5\},\{1,3,4\}, & \{3,4,5\},\{1,2\}, \\ \{3,5\},\{1,2,4\}, & \{1,3,5\},\{2,4\}, \\ \{4,5\},\{1,2,3\}, & \{2,4,5\},\{1,3\}, \\ & \{1,4,5\},\{2,3\}, \\ & \{2,3,5\},\{1,4\}. \end{array}$$

(i) Explain why $f(n,k) = 0$ if $k > n/2$.

(ii) What is the value of $f(n,1)$ and why?

(iii) In forming an allowable partition of $\{1,2,\ldots,n+1\}$ into subsets of at least two elements, we can either

- pair $n+1$ with one other element, leaving $n-1$ elements to deal with, or
- take an allowable partition of $\{1,2,\ldots,n\}$ and add $n+1$ to one of the existing subsets, making a subset of size three or more.

Use this observation to find an equation for $f(n+1,k)$ in terms of $f(n-1,k-1)$ and $f(n,k)$ that holds when $2 \leqslant k < n$.

7. (a) Show that there cannot exist three prime numbers, each greater than 3, which are in arithmetic progression with a common difference less than 5.

(b) Let $k > 3$ be an integer. Show that it is not possible for k prime numbers, each greater than k, to be in in arithmetic progression with a common difference less than or equal to $k+1$.

8. The format for car license plates in a small country is two digits followed by three vowels, e.g. 04 IOU. A license plate is called "confusing" if the digit 0 (zero) and the vowel O are both present on it. For example 04 IOU is confusing but 20 AEI is not. (i) How many distinct number plates are possible in all? (ii) How many of these are *not* confusing?

4. Suppose in a competition 11 matches are to be played, each having one of 3 distinct outcomes as possibilities. The number of ways one can predict the outcomes of all 11 matches such that exactly 6 of the predictions turn out to be correct is

(A) $\binom{11}{6} \times 2^5$ (B) $\binom{11}{6}$ (C) 3^6 (D) none of the above.

5. A set contains $2n+1$ elements. The number of subsets of the set which contain at most n elements is

(A) 2^n (B) 2^{n+1} (C) 2^{n-1} (D) 2^{2n}.

10 In how many ways can 20 identical chocolates be distributed among 8 students so that each student gets at least one chocolate and exactly two students get at least two chocolates each?

(A) 308 (B) 364 (C) 616 (D) $\binom{8}{2}\binom{17}{7}$

6. A club with x members is organized into four committees such that

(a) each member is in exactly two committees,

(b) any two committees have exactly one member in common.

Then x has

(A) exactly two values both between 4 and 8

(B) exactly one value and this lies between 4 and 8

(C) exactly two values both between 8 and 16

(D) exactly one value and this lies between 8 and 16.

18 Let N be a number such that whenever you take N consecutive positive integers, at least one of them is coprime to 374. What is the smallest possible value of N?

(A) 4 (B) 5 (C) 6 (D) 7

7. Let X be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define the set $\mathcal{R}$ by

$$\mathcal{R} = \{(x, y) \in X \times X : x \text{ and } y \text{ have the same remainder when divided by } 3\}.$$

Then the number of elements in $\mathcal{R}$ is

(A) 40 (B) 36 (C) 34 (D) 33.

8. Let A be a set of n elements. The number of ways, we can choose an ordered pair (B, C), where B, C are disjoint subsets of A, equals

(A) n^2 (B) n^3 (C) 2^n (D) 3^n.

11. The number of positive integers which are less than or equal to 1000 and are divisible by none of 17, 19 and 23 equals

(A) 854 (B) 153 (C) 160 (D) none of the above.

19 Let $A_1, A_2, \ldots, A_{18}$ be the vertices of a regular polygon with 18 sides. How many of the triangles $\triangle A_iA_jA_k$, $1 \le i < j < k \le 18$, are isosceles but not equilateral?

(A) 63 (B) 70 (C) 126 (D) 144

21. Consider a sequence of 10 A's and 8 B's placed in a row. By a run we mean one or more letters of the same type placed side by side. Here is an arrangement of 10 A's and 8 B's which contains 4 runs of A and 4 runs of B:

$$A\,A\,A\,B\,B\,A\,B\,B\,B\,A\,A\,B\,A\,A\,A\,A\,B\,B$$

In how many ways can 10 A's and 8 B's be arranged in a row so that there are 4 runs of A and 4 runs of B?

(A) $2\binom{9}{3}\binom{7}{3}$ (B) $\binom{9}{3}\binom{7}{3}$ (C) $\binom{10}{4}\binom{8}{4}$ (D) $\binom{10}{5}\binom{8}{5}$.

6. In a class of 80 students, 40 are girls and 40 are boys. Also, exactly 50 students wear glasses. Then the set of all possible numbers of boys without glasses is

(A) $\{0, \ldots, 30\}$ (B) $\{10, \ldots, 30\}$ (C) $\{0, \ldots, 40\}$ (D) none of these

22. The five vowels—A, E, I, O, U—along with 15 X's are to be arranged in a row such that no X is at an extreme position. Also, between any two vowels there must be at least 3 X's. The number of ways in which this can be done is

(A) 1200 (B) 1800 (C) 2400 (D) 3000

26. Let n be the number of ways in which 5 men and 7 women can stand in a queue such that all the women stand consecutively. Let m be the number of ways in which the same 12 persons can stand in a queue such that exactly 6 women stand consecutively. Then the value of $\frac{m}{n}$ is

(A) 5 (B) 7 (C) $\frac{5}{7}$ (D) $\frac{7}{5}$

Take r such that $1 \leq r \leq n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = \frac{n+1}{r+1}.$$

10. The chance of a student getting admitted to colleges A and B are 60% and 40%, respectively. Assume that the colleges admit students independently. If the student is told that he has been admitted to at least one of these colleges, what is the probability that he has got admitted to college A?

(A) 3/5 (B) 5/7 (C) 10/13 (D) 15/19.

11. Given a positive integer m, we define $f(m)$ as the highest power of 2 that divides m. If n is a prime number greater than 3, then

(A) $f(n^3 - 1) = f(n - 1)$

(B) $f(n^3 - 1) = f(n - 1) + 1$

(C) $f(n^3 - 1) = 2f(n - 1)$

(D) none of the above is necessarily true.

21. Suppose that the number plate of a vehicle contains two vowels followed by four digits. However to avoid confusion, the letter 'O' and the digit '0' are not used in the same number plate. How many such number plates can be formed?

(A) 164025 (B) 190951 (C) 194976 (D) 219049.

27. A general election is to be scheduled on 5 days in May such that it is not scheduled on two consecutive days. In how many ways can the 5 days be chosen to hold the election?

(A) $\binom{26}{5}$ (B) $\binom{27}{5}$ (C) $\binom{30}{5}$ (D) $\binom{31}{5}$.

17. The number of words that can be constructed using 10 letters of the English alphabet such that all five vowels appear exactly once in the word is

(A) $^{21}C_5\ (5!)^2$ (B) $^{21}C_5\ 10!$

(C) $^{10}P_5\ (21)^5$ (D) $^{10}P_5\ ^{21}P_5$

19. Let x, y, z be three natural numbers. Then the number of triplets (x, y, z) such that $xyz = 100$ is

(A) 72 (B) 36 (C) 25 (D) 18

20. How many distinct straight lines can one form that are given by an equation $ax + by = 0$, where a and b are numbers from the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$?

(A) 63 (B) 57 (C) 37 (D) 49

A is a set containing n elements. A subsets P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1. Next, a subset P_2 of A is chosen and again the set is reconstructed by replacing the elements of P_2. In this way, $m(> 1)$ subsets, $P_1, P_2, ..., P_m$ of are chosen. The number of ways of choosing $P_1, P_2, ..., .P_m$ is

A $(2^m - 1)^n$ if $P_1 \cap P_2 \cap ... \cap P_m = \phi$

B 2^{mn} if $P_1 \cup P_2 \cup ... \cup P_m = A$

C 2^{mn} if $P_1 \cap P_2 \cap ... \cap P_m = \phi$

D $(2^m - 1)^n$ if $P_1 \cup P_2 \cup ... \cup P_m = A$

5 balls are to be placed in 3 boxes. Each box can hold all 5 balls. In how many different ways can we place the balls if,

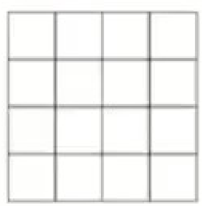
(i) balls & boxes are different

(ii) balls are identical but boxes are different

(iii) balls are different but boxes are identical

(iv) balls as well as boxes are identical

2. The number of squares in the following figure is



(A) 25 (B) 26 (C) 29 (D) 30.

9. Consider all non-empty subsets of the set $\{1, 2, \ldots, n\}$. For every such subset, we find the product of the reciprocals of each of its elements. Denote the sum of all these products as S_n. For example,

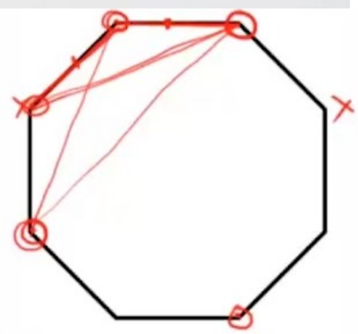
$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{1.2.3}.$$

(a) Show that $S_n = \frac{1}{n} + (1 + \frac{1}{n})S_{n-1}$.

(b) Hence or otherwise, deduce that $S_n = n$.

Permutation and Combination

Q. 'n' lines, no two of which are parallel and no '3' are concurrent. If their points of intersection are joined, then show that number of fresh lines thus formed is $\frac{n(n-1)(n-2)(n-3)}{8}$



(i) Total no of Δ's.

(ii) total no of Δs having exactly one side common

(iii) total no of Δ's having exactly two sides common =

(iv) No of Δs which do not have a side common with the oct

3. Suppose Roger has 4 identical green tennis balls and 5 identical red tennis balls. In how many ways can Roger arrange these 9 balls in a line so that no two green balls are next to each other and no three red balls are together?

(A) 8 (B) 9 (C) 11 (D) 12

28. Two rows of n chairs, facing each other, are laid out. The number of different ways that n couples can sit on these chairs such that each person sits directly opposite to his/her partner is

(A) $n!$ (B) $n!/2$ (C) $2^n n!$ (D) $2n!$.

9. An up-right path is a sequence of points $\mathbf{a}_0 = (x_0, y_0), \mathbf{a}_1 = (x_1, y_1), \mathbf{a}_2 = (x_2, y_2), \ldots$ such that $\mathbf{a}_{i+1} - \mathbf{a}_i$ is either $(1, 0)$ or $(0, 1)$. The number of up-right paths from $(0, 0)$ to $(100, 100)$ which pass through $(1, 2)$ is:

(A) $3 \cdot \binom{197}{99}$ (B) $3 \cdot \binom{100}{50}$ (C) $2 \cdot \binom{197}{98}$ (D) $3 \cdot \binom{197}{100}$.

20. If the word PERMUTE is permuted in all possible ways and the different resulting words are written down in alphabetical order (also known as dictionary order), irrespective of whether the word has meaning or not, then the 720^{th} word would be:

(A) EEMPRTU (B) EUTRPME (C) UTRPMEE (D) MEETPUR.

6. A number is called a palindrome if it reads the same backward or forward. For example, 112211 is a palindrome. How many 6-digit palindromes are divisible by 495?

(A) 10 (B) 11 (C) 30 (D) 45

No. of ONTO - $f: A \to B$ $n(A)=4$; $n(B)=2$

16. Let $A = \{a_1, a_2, \ldots, a_{10}\}$ and $B = \{1, 2\}$. The number of functions $f : A \to B$ for which the sum $f(a_1) + \cdots + f(a_{10})$ is an even number, is

(A) 768 (B) 512 (C) 256 (D) 128

14 Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{a, b, c, d, e\}$. How many functions $f : A \to B$ are there such that for every $x \in A$, there is one and exactly one $y \in A$ with $y \neq x$ and $f(x) = f(y)$?

(A) 450 (B) 540 (C) 900 (D) 5400.

5 balls are to be placed in 3 boxes. Each box can hold all 5 balls. In how many different ways can we place the balls so that no box remains empty if,

(i) balls & boxes are different
(ii) balls are identical but boxes are different
(iii) balls are different but boxes are identical
(iv) balls as well as boxes are identical
(v) balls as well as boxes are identical but boxes are kept in a row.

Q No. of positive integral soln of the inequality:

$$3x + y + z \leq 30.$$

24 What is the number of ordered triplets (a, b, c), where a, b, c are positive integers (not necessarily distinct), such that $abc = 1000$?

(A) 64 (B) 100 (C) 200 (D) 560

An examination has 20 questions. For each question the marks that can be obtained are either -1 or 0 or 4. If S be the set of possible total marks that a student can score in an examination, then find the total number of elements in set S?

5 balls are to be placed in 3 boxes. Each box can hold all 5 balls. In how many different ways can we place the balls if,

(i) balls & boxes are different
(ii) balls are identical but boxes are different
(iii) balls are different but boxes are identical
(iv) balls as well as boxes are identical

7. Let $S = \{1, 2, \ldots, l\}$. For every non-empty subset A of S, let $m(A)$ denote the maximum element of A. Then, show that

$$\sum m(A) = (l-1)2^l + 1$$

where the summation in the left hand side of the above equation is taken over all non-empty subsets A of S.

4 Using only the digits 2, 3 and 9, how many six digit numbers can be formed which are divisible by 6?

(A) 41 (B) 80 (C) 81 (D) 161

(7) Let $A = \{1, 2, \ldots, n\}$. For a permutation $P = (P(1), P(2), \cdots, P(n))$ of the elements of A, let $P(1)$ denote the first element of P. Find the number of all such permutations P so that for all $i, j \in A$:

- if $i < j < P(1)$, then j appears before i in P; and
- if $P(1) < i < j$, then i appears before j in P.

3. You are told that $n = 110179$ is the product of two primes p and q. The number of positive integers less than n that are relatively prime to n (i.e. those m such that $gcd(m, n) = 1$) is 109480. Write the value of $p + q$. Then write the values of p and q.

4. A *step* starting at a point P in the XY-plane consists of moving by *one unit* from P in one of three directions: directly to the right or in the direction of one of the two rays that make the angle of $\pm 120°$ with positive X-axis. (An opposite move, i.e. to the left/southeast/northeast, is not allowed.) A *path* consists of a number of such steps, each new step starting where the previous step ended. Points and steps in a path may repeat.

   Find the number of paths starting at (1,0) and ending at (2,0) that consist of

   (i) exactly 6 steps (ii) exactly 7 steps.

3. 10 mangoes are to be placed in 5 distinct boxes labeled U, V, W, X, Y. A box may contain any number of mangoes including no mangoes or all the mangoes. What is the number of ways to distribute the mangoes so that exactly two of the boxes contain exactly two mangoes each?

2. Consider the equation $x^2 + y^2 = 2007$. How many solutions (x, y) exist such that x and y are positive integers?

   (A) None

   (B) Exactly two

   (C) More than two but finitely many

   (D) Infinitely many.

8. How many non-congruent triangles are there with *integer* lengths $a \le b \le c$ such that $a + b + c = 20$?

26. Two distinct numbers are selected from the set $[1, 2, 3, \ldots, 3n]$. The number of ways in which this can be done, if the sum of the selected numbers is divisible by 3, is

   (a) $\frac{3n(3n-1)}{2}$

   (b) $\frac{n(3n-1)}{2}$

   (c) $\frac{3n(n-1)}{2}$

   (d) None of the above

21. The number of positive integral solutions of the equation $x^2 - y^2 = 3906$ is

   (a) 2

   (b) 1

   (c) 0

   (d) None of the above

4. The number of common terms in the two sequences $\{3, 7, 11, \ldots, 407\}$ and $\{2, 9, 16, \ldots, 709\}$ is

(A) 13 (B) 14 (C) 15 (D) 16.

1. For a natural number m, define $\Phi_1(m)$ to be the number of divisors of m and for $k \geq 2$ define $\Phi_k(m) := \Phi_1(\Phi_{k-1}(m))$. For example, $\Phi_2(12) = \Phi_1(6) = 4$. Find the minimum k such that

$$\Phi_k(2019^{2019}) = 2.$$

5. One needs to choose six real numbers $x_1, x_2, \ldots, x_6$ such that the product of any five of them is equal to other number. The number of such choices is

(A) 3 (B) 33 (C) 63 (D) 93.

1. For a natural number n denote by $\mathrm{Map}(n)$ the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. For $f, g \in \mathrm{Map}(n)$, $f \circ g$ denotes the function in $\mathrm{Map}(n)$ that sends x to $f(g(x))$. [10 marks]

   (a) Let $f \in \mathrm{Map}(n)$. If for all $x \in \{1, \ldots, n\}$ $f(x) \neq x$, show that $f \circ f \neq f$.

   (b) Count the number of functions $f \in \mathrm{Map}(n)$ such that $f \circ f = f$.

7. The greatest common divisor of all numbers of the form $p^2 - 1$, where $p \geq 7$ is a prime, is

(A) 6 (B) 12 (C) 24 (D) 48.

10. A new flag of ISI club is to be designed with 5 vertical strips using some or all of the four colours: green, maroon, red and yellow. In how many ways this can be done so that no two adjacent strips have the same colour ?

(A) 120 (B) 324 (C) 432 (D) 576.

1. The highest power of 7 that divides 100! is

(A) 14 (B) 15 (C) 16 (D) 18.

A flexadecimal number consists of a sequence of digits, with the rule that the rightmost digit must be 0 or 1, the digit to the left of it is 0, 1, or 2, the third digit (counting from the right) must be at most 3, and so on. As usual, we may omit leading digits if they are zero. We write flexadecimal numbers in angle brackets to distinguish them from ordinary, decimal numbers. Thus $\langle 34101\rangle$ is a flexadecimal number, but $\langle 231\rangle$ is not, because the digit 3 is too big for its place. (If flexadecimal numbers get very long, we will need 'digits' with a value more than 9.)

The number 1 is represented by $\langle 1\rangle$ in flexadecimal. To add 1 to a flexadecimal number, work from right to left. If the rightmost digit d_1 is 0, replace it by 1 and finish. Otherwise, replace d_1 by 0 and examine the digit d_2 to its left, appending a zero at the left if needed at any stage. If $d_2 < 2$, then increase it by 1 and finish, but if $d_2 = 2$, then replace it by 0, and again move to the left. The process stops when it reaches a digit that can be increased without becoming too large. Thus, the numbers 1 to 4 are represented as $\langle 1\rangle$, $\langle 10\rangle$, $\langle 11\rangle$, $\langle 20\rangle$.

(i) Write the numbers from 5 to 13 in flexadecimal.

(ii) Describe a workable procedure for converting flexadecimal numbers to decimal, and explain why it works. Demonstrate your procedure by converting $\langle 1221\rangle$ to decimal.

2. The number of 6-digit positive integers whose sum of the digits is at least 52 is

(A) 21 (B) 22 (C) 27 (D) 28.

3. The sum of all 3-digit numbers that leave a remainder of 2 when divided by 3 is

(A) 189700 (B) 164850 (C) 164750 (D) 149700.

4. Suppose that 6-digit numbers are formed using each of the digits $1, 2, 3, 7, 8, 9$ exactly once. The number of such 6-digit numbers that are divisible by 6 but not divisible by 9 is equal to

(A) 120 (B) 180 (C) 240 (D) 360.

3. (a) Show that there are exactly 2 numbers a in $\{2, 3, \ldots, 9999\}$ for which $a^2 - a$ is divisible by 10000. Find these values of a.

(b) Let n be a positive integer. For how many numbers a in $\{2, 3, \ldots, n^2 - 1\}$ is $a^2 - a$ divisible by n^2 ? State your answer suitably in terms of n and justify.

6. Find all pairs (p, n) of positive integers where p is a prime number and $p^3 - p = n^7 - n^3$.

1 Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two permutations of the numbers $1, 2, \cdots, n$. Show that

$$\sum_{i=1}^{n} i(n+1-i) \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} i^2$$

The number of ways in which 12 identical balls can be grouped into four marked non-empty boxes A, B, C, D so that n(A) < n(B).

2 Let a, b, c, d be distinct digits such that the product of the 2-digit numbers $\overline{ab}$ and $\overline{cb}$ is of the form $\overline{ddd}$. Find all possible values of $a+b+c+d$.

10 There are 100 people in a queue waiting to enter a hall. The hall has exactly 100 seats numbered from 1 to 100. The first person in the queue enters the hall, chooses any seat and sits there. The n-th person in the queue, where n can be $2, \ldots, 100$, enters the hall after $(n-1)$-th person is seated. He sits in seat number n if he finds it vacant; otherwise he takes any unoccupied seat. Find the total number of ways in which 100 seats can be filled up, provided the 100-th person occupies seat number 100.

In how many ways we can distribute 5 different balls in 4 different boxes so that no box is empty.

In how many different ways we can buy an icecream for 10 €. if we have 3 notes of 5 €, 6 notes of 2 €, 2 notes of 1 €.

3. Write the set of all positive integers in triangular array as

$$\begin{array}{cccccccc} 1 & 3 & 6 & 10 & 15 & \cdot & \cdot \\ 2 & 5 & 9 & 14 & \cdot & \cdot & \cdot \\ 4 & 8 & 13 & \cdot & \cdot & \cdot & \cdot \\ 7 & 12 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 11 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Find the row number and column number where 20096 occurs. For example 8 appears in the third row and second column.

20. Consider six players P_1, P_2, P_3, P_4, P_5 and P_6. A team consists of two players. (Thus, there are 15 distinct teams.) Two teams play a match exactly once if there is no common player. For example, team $\{P_1, P_2\}$ can not play with $\{P_2, P_3\}$ but will play with $\{P_4, P_5\}$. Then the total number of possible matches is

(A) 36 (B) 40 (C) 45 (D) 54.