If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$$\int_0^1 (\sin x + a\cos x)^3 dx - \frac{4a}{\pi-2}\int_0^1 x\cos x\, dx = 2$$

$$\int_0^{\pi/4} \frac{x\,dx}{\cos x(\cos x+\sin x)}$$ then find the value of $1000(a_1^2+a_2^2+a_3^2)$.

$$\int_0^{4/\pi}\left(3x^2 \sin\frac{1}{x} - x.\cos\frac{1}{x}\right)dx$$ has the value :

(A) $\frac{8\sqrt{2}}{\pi^3}$ (B) $\frac{24\sqrt{2}}{\pi^3}$ (C*) $\frac{32\sqrt{2}}{\pi^3}$ (D) None

Let $f(x) = \frac{\sin x}{x}$, then $\int_0^{\pi/2} f(x) f\left(\frac{\pi}{2}-x\right)dx =$

(A*) $\frac{2}{\pi}\int_0^{\pi} f(x)dx$ (B) $\int_0^{\pi} f(x)dx$ (C) $\pi\int_0^{\pi} f(x)dx$ (D) $\frac{1}{\pi}\int_0^{\pi} f(x)dx$

2. If $I_1 = \int_{-100}^{101} \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ and $I_2 = \int_{-100}^{101} \frac{dx}{5+2x-2x^2}$, then $\frac{I_1}{I_2}$ is

(A) $\frac{1}{2}$ (B) 2

(C) 0 (D) none of these

If $I_n = \int_0^{\infty} e^{-x}(\sin x)^n dx$ $(n > 1)$, then the value of $\frac{101\, I_{10}}{10\, I_8}$ is equal to __________.

Let $g(x) = f_n(x)$, where $f_2(x) = f(f(x))$, $f_3(x) = f(f(f(x)))$ ..... and $f(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$ then $\int_{1/4}^{3/4} g(x)\,dx$ is

(A) $\frac{n}{2}+2$ (B) $2^n + 1$

(C) $\frac{1}{4}$ (D) $\frac{3}{4}$

$\int_0^{102}(x-1)(x-2)......(x-100)\left(\frac{1}{x-1}+\frac{1}{x-2}+.....\frac{1}{x-100}\right)dx = \underline{|p} - \underline{|q}$ then p + q is equal to

(A) 203 (B) 201

(C) 202 (D) 204

Let $f:R^+ \to R$ be a differentiable function with $f(1)=3$ and satisfying :

$$\int_1^{xy} f(t)\,dt = y\int_1^{x} f(t)\,dt + x\int_1^{y} f(t)\,dt \ \forall\ x,y \in R^+$$, then $f(e) =$

(a) 3 (b) 4 (c) 1 (d) None of these

If $\int_0^1 \left(\sum_{r=1}^{2013} \frac{x}{x^2+r^2}\right)\left(\prod_{r=1}^{2013}(x^2+r^2)\right)dx = \frac{1}{2}\left[\left(\prod_{r=1}^{2013}(1+r^2)\right) - k^2\right]$

then $k =$

(a) 2013 (b) 2013! (c) 2013^2 (d) 20

If $A_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}dx;\ B_n = \int_0^{\pi/2}\left(\frac{\sin nx}{\sin x}\right)^2 dx$, for $n \in N$, then

a) $A_{n+1} = A_n$ b) $B_{n+1} = B_n$ c) $A_{n+1} - A_n = B_{n+1}$ d) $B_{n+1} - B_n = A_{n+1}$

$$\int_0^{\frac{\pi}{4}} \left(\frac{x}{x\sin x + \cos x}\right)^2 dx =$$

a) $\frac{5-\pi}{5+\pi}$ b) $\frac{2}{4+\pi}$ c) $\frac{4-\pi}{4+\pi}$ d) $\frac{4+\pi}{4-\pi}$

The absolute value of $\frac{\int_0^{\pi/2}(x\cos x + 1)e^{\sin x}dx}{\int_0^{\pi/2}(x\sin x - 1)e^{\cos x}dx}$ is equal to :

(a) e (b) πe (c) $e/2$ (d) π/e

The value of definite integral $\int_{-\pi}^{\pi} \frac{x^2}{1+\sin x+\sqrt{1+\sin^2 x}}dx$, is :

(a) $\frac{3\pi^3}{2}$ (b) $\frac{\pi^3}{3}$ (c) $\frac{2\pi^3}{3}$ (d) $\frac{\pi^3}{6}$

The value of definite integral $\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2 - x + 1} dx$ is equal to :

(a) $\frac{\pi^2}{6\sqrt{3}}$ (b) $\frac{\pi^2}{3\sqrt{3}}$ (c) $\frac{\pi^2}{12\sqrt{3}}$ (d) $\frac{\pi^2}{4\sqrt{3}}$

4

Let $V_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$. If $\sum_{r=1}^{100} V_r = \frac{k\pi}{2}$, where $k \in N$, then k is equal to :

(a) 100 (b) 2525 (c) 5050 (d) 4950

Let $f(x)$ be a continuous and periodic function such that $f(x) = f(x + T)$ for all $x \in R, T > 0$. If $\int_{-2T}^{a+5T} f(x)\,dx = 19\ (a > 0)$ and $\int_0^T f(x)\,dx = 2$, then $\int_0^a f(x)\,dx$ is equal to :

(a) 3 (b) 5 (c) 7 (d) 9

If x and y are independent variables and $f(x) = \left(\int_0^x e^{x-y} f'(y)\,dy\right) - (x^2 - x + 1)\,e^x$, where $f(x)$ is a differentiable function, then $f\left(\frac{-1}{2}\right)$ equals :

(a) $2\sqrt{e}$ (b) $-2\sqrt{e}$ (c) $\frac{-2}{\sqrt{e}}$ (d) $\frac{2}{\sqrt{e}}$

Paragraph for Question Nos. 19 and 20

Let a function 'f' satisfies $f(-x) = f(x)$ and $f(3 + x) = f(1 - x)\ \forall\ x \in R$ and

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - 2x, & 1 < x \le 2 \end{cases}.$$

19. The number of points where $f(x)$ is discontinuous in [0, 100], is :

(a) 100 (b) 50 (c) 25 (d) 0

20. The value of $\int_0^{100} f(x)\,dx$ is equal to :

(a) – 75 (b) – 50 (c) – 25 (d) 0

Paragraph for Question Nos. 31 and 32

Let $f_n(x) = \sum_{n=1}^{n} \frac{\sin^2 x}{\cos^2\left(\frac{x}{2}\right) - \cos^2\left(\frac{2n+1}{2}\right)x}$ and let $g_n(x) = \prod_{n=1}^{n} f_n(x)$

$\forall\, n = 1, 2, 3, \ldots\ldots\ldots\ldots$

31. Let $I_n = \int_0^{\pi} \frac{f_n(x)}{g_n(x)}\, dx$. If $\sum_{k=1}^{100} I_n = k\pi$, then the value of k is :

(a) 50 (b) 25 (c) 100 (d) 75

32. The value of $\left[\lim_{x\to 0} \int_0^{x} \frac{9\, dt}{x f_9(t)\, g_9(t)}\right]$ is :

(a) 50 (b) 10 (c) 100 (d) 25

Let f be a differentiable function defined such that $f: [0, 27] \to \left[\frac{1}{3}, 6\right]$ and $f'(x) < 0$ $\forall\, x \in D_f$. If $\int_0^{27} x f'(x)\, dx = \lambda - 3\int_0^{3} x^2 f(x^3)\, dx$ then find the value of λ.

Let $g(x)$ be a real valued function defined on the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ such that $g(x) = e^{2x} + \int_0^{\sin x} \frac{e^t}{\cos^2 x + 2t \sin x - t^2}\, dt\ \forall\, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Also $f(x)$ be the inverse function of $g(x)$, where $0 \le x \le \frac{\pi}{2}$. Find the value of $\frac{1}{(f'(1))^2} + g(0) + g'(0) + g''(0)$.

The value of

$$I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + (x + 1) \cos (x + 1)]\, dx, \text{ is}$$

(a) − 4 (b) − 3

(c) − 2 (d) − 1

Ans. (c)

If $I =$

$$\int_0^{\pi} e^{|(1/2)\cos x|}\left\{2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right\}\sin x \; dx,$$

then I equals

(a) $7\sqrt{e}\cos(1/2)$ (b) $7\sqrt{e}\,[\cos(1/2) - \sin(1/2)]$

(c) 0 (d) none of these

Ans. (d)

Example 20 The natural number n (≤ 5) for which

$$I_n = \int_0^1 e^x (x-1)^n \, dx = 16 - 6e$$

is

(a) 2 (b) 3

(c) 4 (d) 5

Ans. (b)

If $I_1 = \int_0^{\pi/2} f(\sin 2x) \sin x \, dx$

and $I_2 = \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$,

then $\dfrac{I_1}{I_2}$ equals

(a) 1 (b) $1/\sqrt{2}$

(c) $\sqrt{2}$ (d) 2

Ans. (c)

Example If $I = \int_{-1}^{2} |x \sin \pi x| \, dx$, then I equals

(a) $1/\pi$ (b) $2/\pi$

(c) $4/\pi$ (d) $5/\pi$

Ans. (d)

Example 43 If $I = \int_{\alpha}^{\beta}\left[\log\log x + \frac{1}{(\log x)^2}\right] dx$, then I equals

(a) $\alpha \log\log \alpha - \beta \log\log \beta$

(b) $\frac{1}{\alpha} - \frac{1}{\beta} + \log\log \alpha - \log\log \beta$

(c) $\frac{\beta - \alpha}{\alpha\beta} + \alpha \log\log \alpha - \beta \log\log \beta$

(d) none of these

Ans. (d)

Example The value of $\int_0^{\pi} \frac{x \sin 2x \sin((\pi/2)\cos x)}{2x - \pi} dx$ is

(a) $4/\pi^2$ (b) $8/\pi^2$

(c) $2\int_0^1 t \sin(\pi/2)\, t)\, dt$ (d) $2/\pi^2$

Ans. (b), (c)

Example 64 The value of $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ is

(a) $\left(\int_{\pi}^{5\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx\right)^2$ (b) $\pi^2/16$

(c) $3\pi^2/4$ (d) $\pi^2/2$

Show that

$$\int_0^{\pi/2} f(\sin 2x) \sin x \, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$$

[1990]

Prove that for any positive integer k,

$$\frac{\sin 2kx}{\sin x} = 2 [\cos x + \cos 3x + \ldots + \cos (2k-1)x].$$

Hence, prove that

$$\int_0^{\pi/2} \sin 2kx \cot x \, dx = \frac{\pi}{2}$$

Show that

$$\int_0^{n\pi+V} |\sin x| \, dx = 2n + 1 - \cos V$$

where n is a positive integer and $0 \leq V < \pi$.

For $n > 0$

$$\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx = \underline{\qquad\qquad}.$$

If $g(x) = \int_0^x \cos^4 t \, dt$, $g(x + \pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$

(c) $g(x)\, g(\pi)$ (d) $\dfrac{g(x)}{g(\pi)}$ [1

65. Let $f(x)$ be a non constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'\left(\frac{1}{4}\right) = 0$ Then

(a) $f'(x)$ vanishes at least twice on $[0, 1]$

(b) $f'\left(\frac{1}{2}\right) = 0$

(c) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$

(d) $\int_0^{1/2} f(t) e^{\sin \pi t} \, dt = \int_{1/2}^{1} f(1-t) e^{\sin \pi t} \, dt$

Consider the function defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly define a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly define a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

[2008]

67. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f'(-10\sqrt{2})$

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$

(c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

68. The area of the region bounded by the curves $y = f(x)$, the x-axis, and the lines x = a and $x = b$, where $-\infty < a < b < -2$, is

(a) $\int_a^a \frac{x}{3(f(x)^2)-1)} dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3((f(x)^2)-1)} dx + bf(b)$ – bf(b) + af(a)

(c) $\int_a^b \frac{x}{3((f(x)^2)-1)} dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3((f(x)^2)-1)} dx - bf(b) + af(a)$

69. $\int_{-1}^{1} g'(x)dx =$

(a) $2g(-1)$ (b) 0

(c) $-2g(1)$ (d) $2g(1)$

Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x+T) = f(x)$.

If $I = \int_0^T f(x)dx$ then the value of $\int_3^{3+3T} f(2x)dx$ is

(a) 3/2*I* (b) 2*I* (c) 3*I* (d) 6*I*

Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$, for all $x \in (-1,1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

(2010)

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{e}$

The value of $\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ell n6 - x^2)} dx$ is **(2011)**

(a) $\frac{1}{4}\ell n\frac{3}{2}$ (b) $\frac{1}{2}\ell n\frac{3}{2}$ (c) $\ell n\frac{3}{2}$ (d) $\frac{1}{6}\ell n\frac{3}{2}$

For $x>0$, let $f(x) = \int_e^x \frac{\ln t}{1+t} dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

The value of $5050 \frac{\int_0^1 (1-x^{50})^{100} dx}{\int_0^1 (1-x^{50})^{101} dx}$ is.

For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd,} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$ is

Find the minimum value of the function

$$f(x) = \int_0^2 |x - t| \, dt.$$

If $\int_0^{\pi/4} \frac{\ln(\cot x)}{\left((\sin x)^{2009} + (\cos x)^{2009}\right)^2} \cdot (\sin 2x)^{2008} \, dx$

$= \frac{a^b \ln a}{c^2}$ (where a, b, c are in their lowest terms)

then find the value of $(a + b + c)$.

Prove that $\int \sin n\theta \sec \theta \, d\theta$

$$= -\frac{2 \cos(n-1)\theta}{n-1} - \int \sin(n-2)\theta \sec \theta \, d\theta.$$

Hence or otherwise

evaluate $\int_0^{\pi/2} \frac{\cos 5\theta \sin 3\theta}{\cos \theta} \, d\theta$.

If $U_n = \int_0^{\pi} \left(\frac{1 - x \cos nx}{1 - \cos x} \right) dx$ where n is a positive integer or zero, then show that $U_{n+2} + U_n = 2U_{n+1}$. Hence show that $\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta} = \frac{n\pi}{2}$.

$$\text{If } I_n = \int_{-\infty}^{0} e^x \sin^n x\,dx \ \forall\, n \geq 2 \in N,$$

then prove that I_{n-2}, I_n, I_{n+2} cannot be in G.P.

Prove that

$$\int_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} = 1$$

Find the value of

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$$

If $\int_0^1 \frac{e^t\,dt}{t+1} = a$, then show that

$\int_{b-1}^{b} \frac{e^{-t}\,dt}{t-b-1}$ is equal to $-ae^{-b}$.

Evaluate

$$I = \int_0^{\pi/4} \frac{x^2(\sin 2x - \cos 2x)}{(1+\sin 2x)\cos^2 x}\,dx$$

If $\int_0^{\pi}\left(\frac{x}{1+\sin x}\right)^2 dx=\lambda$ then show that $\int_0^{\pi}\frac{2x^2\cos^2(x/2)}{(1+\sin x)^2}dx=\lambda+2\pi-\pi^2$.

Find the value of the definite integral

$\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)dx$.

For any $t \in R$ and f being a continuous function,

let $I_1=\int_{\sin^2 t}^{1+\cos^2 t} x f(x(2-x))\,dx$ and

$I_2=\int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\,dx$ then find $\frac{I_1}{I_2}$.

Suppose $I_1=\int_0^{\pi/2}\cos(\pi\sin^2 x)dx$,

$I_2=\int_0^{\pi/2}\cos(2\pi\sin^2 x)dx$ and

$I_3=\int_0^{\pi/2}\cos(\pi\sin x)dx$, then show that

$I_1=0$ and $I_2+I_3=0$.

$\int_0^{\pi/2}\ln\sin x dx=\int_0^{\pi/2}\ln\cos x\,dx=-\frac{\pi}{2}\ln 2$ Evaluate $\int_{-\pi/4}^{\pi/4}\ln(\sin x+\cos x)\,dx$.

Evaluate $I=\int_0^1\frac{\ln(1+x)}{1+x^2}dx$

Find the value of

$$2^{2010}\frac{\int_0^1 x^{1004}(1-x)^{1004}\,dx}{\int_0^1 x^{1004}(1-x^{2010})^{1004}\,dx}.$$

Find the value of

$$\int_0^{2n\pi} \max.(\sin x, \sin^{-1}(\sin x))\,dx \text{ (where } n \in I).$$

Let $F(x)$ be a non-negative continuous function defined on R such that $F(x) + F\left(x+\frac{1}{2}\right) = 3$.

Find the value of $\int_0^{1500} F(x)dx$.

For $\theta \in \left(0, \frac{\pi}{2}\right)$, find the value of

$$\int_0^{\theta} \ln(1+\tan\theta \tan x)\,dx.$$

Q. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx, n = 0, 1, 2, \ldots,$ then– (A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$ (C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ for $n = 1, 2, 3, \ldots$. Then,

A $S_n < \frac{\pi}{3\sqrt{3}}$ B $S_n > \frac{\pi}{3\sqrt{3}}$ C $T_n < \frac{\pi}{3\sqrt{3}}$ D $T_n > \frac{\pi}{3\sqrt{3}}$

If $f(x) = x + \int_0^1 [xy^2 + x^2y]\, f(y)\,dy$ where x and y are independent variable. Find f(x).

If $f(x) = \int_{\pi^2/16}^{x^2} \frac{\sin x \cdot \sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}} . d\theta$ then the value of $f'\left(\frac{\pi}{2}\right)$, is

(A*) π (B) $-\pi$ (C) 2π (D) 0

$$I = \int_0^2 [x + [x + [x]]]\,dx$$

The value of $\int_1^a [x] f'(x)\,dx$ is ; $a > 1$

(A) $a f(a) - (f(1) + f(2) + \cdots + f([a]))$

(B) $[a] f(a) - (f(1) + f(2) + \cdots f([a]))$

(C) $[a] f([a]) - (f(1) + f(2) \cdots f(a))$

(D) $a f([a]) - (f(1) + f(2) \cdots f(a))$

General $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 + x^5}} < \frac{\pi}{6}$

Prove the inequalities:

(a) $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}$

(b) $2e^{-1/4} < \int_0^2 e^{x^2 - x}\,dx < 2e^2$.

(c) $a < \int_0^{2\pi} \frac{dx}{10 + 3\cos x} < b$ then find a & b.

(d) $\frac{1}{2} \le \int_0^2 \frac{dx}{2 + x^2} \le \frac{5}{6}$

Show that, $\int_0^{\pi} \left|\frac{\sin nx}{x}\right| dx \ge \frac{2}{\pi}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$

Let A_n be the area bounded by the curve $y = (\tan x)^n$ & the lines $x = 0$, $y = 0$ & $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n-1)$ & deduce that $1/(2n+2) < A_n < 1/(2n-2)$.

17. Let $[x]$ denote the greatest integer less than or equal to x. The value of the integral

$$\int_1^n [x]^{x-[x]}\, dx$$

is equal to

(A) $1+\frac{2^3}{\log_e 2}-\frac{2^2}{\log_e 2}+\frac{3^4}{\log_e 3}-\frac{3^3}{\log_e 3}+\cdots+\frac{(n-1)^n}{\log_e(n-1)}-\frac{(n-1)^{n-1}}{\log_e(n-1)}$.

(B) $1+\frac{1}{\log_e 2}+\frac{2}{\log_e 3}+\cdots+\frac{n-2}{\log_e(n-1)}$.

(C) $\frac{1}{2}+\frac{2^2}{3}+\cdots+\frac{n^{n+1}}{n+1}$.

(D) $\frac{2^3-1}{3}+\frac{3^4-2^3}{4}+\cdots+\frac{n^{n+1}-(n-1)^n}{n+1}$.

If S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, (2012)

(a) $S \geq \frac{1}{e}$ (b) $S \geq 1 - \frac{1}{e}$

(c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \leq \int_{1/2}^{1} f(x)\, dx \leq M$, then the possible values of m and M are (2015 Adv.)

(a) $m = 13, M = 24$

(b) $m = \frac{1}{4}, M = \frac{1}{2}$

(c) $m = -11, M = 0$

(d) $m = 1, M = 12$

48. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)}\,dx$, then (2017 Adv.)

(a) $I > \log_e 99$ (b) $I < \log_e 99$

(c) $I < \frac{49}{50}$ (d) $I > \frac{49}{50}$

Q7

f is a differentiable function, such that $f(f(x)) = x$, where $x \in [0,1]$. Also, $f(0) = 1$. Find the value of

$$\int_0^1 (x - f(x))^{2016}\,dx$$

$f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [1, 3]$ then g(x) has

(A) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$

(B) local maxima at $x = 1$ and local minima at $x = 2$

(C) no local maxima

(D) no local minima

Solve the equation for y as a function of x, satisfying

$$x \cdot \int_0^x y(t)\,dt = (x+1)\int_0^x t \cdot y(t)\,dt, \text{ where } x > 0, \text{ given } y(1) = 1.$$

Prove that $f(x) = \int_2^{e^x} \left(9\cos^2(2\ln t) - 25\cos(2\ln t) + 17\right)dt$ is always an increasing function of x, $\forall\, x \in R$

Find the value of x > 1 for which the function

$$F(x) = \int_{x}^{x^2} \frac{1}{t} \ln\left(\frac{t-1}{32}\right) dt \text{ is increasing and decreasing.}$$

Assuming the validity of differentiation under the integral sign, show that

$$\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\, dx = \pi(\sqrt{1+y}-1), \text{ where } y > -1$$

Evaluate $\int_0^{\infty} (e^{-x}/x)\{a-(1/x)+(1/x)\times e^{-ax}\}dx$

show that $\int_{\pi/2-\alpha}^{\pi/2} \sin\theta \cos^{-1}(\cos\alpha \operatorname{cosec}\theta)\, d\theta = \frac{\pi}{2}(1-\cos\alpha)$

Q.13 Let f be a real valued function defined on the interval $(0, \infty)$ by

$$f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\, dt$$

Then which of the following statements is (are) true?

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$.

Passage Based Questions

Let $f(x) = (1-x)^2 \sin^2 x + x^2$, $\forall\, x \in R$ and
$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\, dt \;\forall\, x \in (1, \infty)$.

15. Consider the statements

P : There exists some $x \in R$ such that, $f(x) + 2x = 2(1 + x^2)$.

Q : There exists some $x \in R$ such that, $2f(x) + 1 = 2x(1 + x)$.

Then,

(a) both P and Q are true (b) P is true and Q is false

(c) P is false and Q is true (d) both P and Q are false

16. Which of the following is true?

(a) g is increasing on $(1, \infty)$

(b) g is decreasing on $(1, \infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

Assuming the validity of differentiation under the integral sign, show that

$$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx = \frac{\pi}{2}\log(1+a), a \geq 0.$$

Also find the value of integral if $a < 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x], & x \leq 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)}\, dx$, then the value of $(4I - 1)$ is

$$\int_0^\infty \left[\frac{2}{e^x}\right] dx$$

If $I_n = \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$, then $I_n = \frac{n-1}{n} \cdot I_{n-2}$ where $n \in N$.

$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....\frac{1}{2}\cdot\frac{\pi}{2}$; n is even= $\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)....\frac{2}{3}$; n is odd.

$\int_0^{\pi/2} Sin^m x \cdot Cos^n x\, dx = \frac{[(m-1)(m-3)(m-5)....][(n-1)(n-3)(n-5)...]}{(m+n)(m+n-2)(m+n-4)(m+n-6).....} \cdot K$ If m and n are both even then $K = \frac{\pi}{2}$, otherwise $K = 1$.

47. Let $f : \left[\frac{1}{2}, 1\right] \to$ R (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval

(A) $(2e - 1, 2e)$
(B) $(e - 1, 2e - 1)$
(C) $\left(\frac{e-1}{2}, e-1\right)$
(D) $\left(0, \frac{e-1}{2}\right)$

Sol. **(D)**

41. For a $\in$ R (the set of all real numbers), $a \neq -1$, $\lim_{n\to\infty} \frac{\left(1^a + 2^a + ... + n^a\right)}{(n+1)^{a-1}\left[(na+1)+(na+2)+...+(na+n)\right]} = \frac{1}{60}$.

Then a =
(A) 5
(B) 7
(C) $\frac{-15}{2}$
(D) $\frac{-17}{2}$

Sol. **(B, D)**

57. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\} dx$ is ___________

46. The following integral $\int_{\pi/4}^{\pi/2} (2\operatorname{cosec} x)^{17}\, dx$ is equal to

(A) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u + e^{-u}\right)^{16} du$
(B) $\int_0^{\log(1+\sqrt{2})} \left(e^u + e^{-u}\right)^{17} du$
(C) $\int_0^{\log(1+\sqrt{2})} \left(e^u - e^{-u}\right)^{17} du$
(D) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u - e^{-u}\right)^{16} du$

Paragraph For Questions 55 and 56

Given that for each $a \in (0, 1)$, $\lim_{h \to 0^+} \int_h^{1-h} t^{-a} (1-t)^{a-1}\, dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$.

55. The value of $g\left(\frac{1}{2}\right)$ is

(A) π (B) 2π

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

56. The value of $g'\left(\frac{1}{2}\right)$ is

(A) $\frac{\pi}{2}$ (B) π

(C) $-\frac{\pi}{2}$ (D) 0

57. Match the following:

	List – I	List – II	
(P)	The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\,dx = 1$, is	(1)	8
(Q)	The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(2)	2
(R)	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)}\,dx$ equals	(3)	4
(S)	$\frac{\left(\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_{0}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	(4)	0

Codes:

	P	Q	R	S
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

41. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in \mathbb{R}$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

47. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)} dx$, then the value of $(4I - 1)$ is

47. If

$$\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$$

where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is

48. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|dt$ for all $x \in [-1, 2]$. If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is

54. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt}{\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt} = L ?$$

(A) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (B) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (D) $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

56. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(A) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ (B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)dx = 1$

PARAGRAPH 1

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

57. The correct statement(s) is(are)

(A) $f'(1) < 0$ (B) $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$ (D) $f'(x) = 0$ for some $x \in (1, 3)$

58. If $\int_1^3 x^2 F'(x)\,dx = -12$ and $\int_1^3 x^3 F''(x)\,dx = 40$, then the correct expression(s) is(are)

(A) $9f'(3) + f'(1) - 32 = 0$ (B) $\int_1^3 f(x)\,dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$ (D) $\int_1^3 f(x)\,dx = -12$

52. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}\,dt = 2x - 1$ is

Sol. **(1)**

41. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x}\,dx$ is equal to

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$

(C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

Sol. **(A)**

44. Let $f(x) = \lim\limits_{n\to\infty} \left(\frac{n^n (x+n)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)} \right)^{\frac{x}{n}}$, for all $x > 0$. Then

(A) $f\left(\frac{1}{2}\right) \geq f(1)$ (B) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

(C) $f'(2) \leq 0$ (D) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

Sol. **(B, C)**

Q.48 Let f: R → R be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If

$$g(x) = \int_x^{\pi/2} \left[f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)\right] dt$$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$

Sol. **2**

Q.5 Let f : R → R and g : R → R be two non-constant differentiable functions. If

$$f'(x) = (e^{(f(x)-g(x))})\, g'(x) \text{ for all } x \in R,$$

and f(1) = g(2) = 1, then which of the following statement(s) is (are) TRUE ?

(A) $f(2) < 1 - \log_e 2$ (B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$ (D) $g(1) < 1 - \log_e 2$

Sol. **B, C**

Q.6 Let f : [0, ∞) → R be a continuous function such that

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$$

for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE ?

(A) The curve y = f(x) passes through the point (1, 2)

(B) The curve y = f(x) passes through the point (2, –1)

(C) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(D) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

Sol. **B, C**

Q.7 The value of the integral $\int_0^{1/2} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{1/4}} dx$ is ______ .

Sol. **2**

Q.2 If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then 27 I^2 equals _____

Sol. **4.00**

Q.4 For $a \in \mathbb{R}\ |a| > 1$, let

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+...+\frac{1}{(an+n)^2}\right)}\right)=54$$

Then the possible value(s) of a is/are

A. 8 B. –6

C. 7 D. –9

Sol. **A, D**

Q.1 The value of the integral

$$\int_0^{\pi/2}\frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta \text{ equals } \_\_\_\_$$

Q.17. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$. If $F: [0, \pi] \to \mathbb{R}$ is defined by $F(x)=\int_0^x f(t)dt$, and if

$$\int_0^{\pi}\left(f'(x)+F(x)\right)\cos x\, dx = 2$$

then the value of $f(0)$ is _____

Sol. **4.00**