If α is nonreal and $\alpha = \sqrt[5]{1}$ then the value of $2^{|1 + \alpha + \alpha^2 + \alpha^{-2} - \alpha^{-1}|}$ is equal t

(A*) 4 (B) 2 (C) 1

Two regular polygons are inscribed in the same circle: the first has 1 731 sides and the second has 4 039. If the two polygons have at least one vertex in common, how many vertices in total will coincide?

Two regular polygons are inscribed in the same circle: the first has 1 731 sides and the second has 4 039. If the two polygons have at least one vertex in common, how many vertices in total will coincide?

Answer

Recall that the nth roots of unity lie at the vertices of a regular n-gon. Hence, we can restate the problem as finding the number of common roots of $z^{1731} - 1 = 0$ and $z^{4039} - 1 = 0$. Using the theorem about the common roots of unity, we know that the common roots will be the roots of $z^d - 1 = 0$ where $d = \gcd(1731, 4039) = 577$. Therefore, if the two polygons have at least one vertex in common, they will have a total of 577 vertices in common.

For how many pairs of real numbers (a, b) does the relation $(a + bi)^{2020} = a - bi$ hold?

For $w = e^{\pi i/11}$ find

$$\prod_{k=0}^{11} \left(w^k - 2w^{-k}\right).$$

Given

$$f(x) = x^{13} + 2x^{12} + 3x^{11} + 4x^{10} + \cdots + 13x + 14,$$

denote

$$N = f(a) \times f\left(a^2\right) \times f\left(a^3\right) \times \cdots \times f\left(a^{14}\right),$$

where $a = \cos\left(\frac{2\pi}{15}\right) + i\sin\left(\frac{2\pi}{15}\right)$.

Then what is the value of M such that $N^{\frac{1}{M}} = 15$?

Find the sum of all the 1000^{th} powers of the 17^{th} roots of unity.

If $\beta \neq 1$ be any n^{th} root of unity then $1 + 3\beta + 5\beta^2 + \ldots\ldots n$ terms equals

1) $\frac{2n}{1-\beta}$ 2) $\frac{-2n}{1-\beta}$ 3) $-\frac{2n}{(1-\beta)^2}$ 4) $\frac{2n}{(1-\beta)^2}$

If $1, x_1, x_2, x_3$ are the roots of $x^4 - 1 = 0$ and ω is a complex cube root of unity, the valu

$\frac{(\omega^2 - x_1)(\omega^2 - x_2)(\omega^2 - x_3)}{(\omega - x_1)(\omega - x_2)(\omega - x_3)}$ *is*

Let n be an integer greater than 7. Let $z^n - 1 = 0$ has roots $1, \lambda_1, \lambda_2, ..., \lambda_{n-1}$ then

A) $\sum_{i=1}^{n-1} \lambda_i = -1$

B) $\sum \sum_{1 \le i < j \le n-1} \lambda_i \lambda_j = 1$

C) $\sum \sum \sum_{1 \le i < j < k \le n-1} \lambda_i \lambda_j \lambda_k = -1$

D) $\lambda_1 \lambda_2 .... \lambda_{n-1} = (-1)^n$

If $n \ge 3, \sum_{r=1}^{n-1} (n-r) \cos \frac{2r\pi}{n} =$

A) n B) n-1 C) $\frac{n}{2}$ D) $-\frac{n}{2}$

The sum $z = \sum_{m=1}^{4n+1} \left\{ \sum_{r=1}^{m-1} e^{i\left(\frac{2r\pi}{m} + \frac{\pi}{2}\right)} \right\}^m$

A) is independent of n B) $-i$

C) $\text{Re}(z) = 1$ D) $\text{Re}(z) = -1$

Problem Show that

$$\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right)^{4\pi}=1648,$$

where p and q are positive integers.

If ω^j, $j = 0, 1, 2, \ldots, n-1$ are the nth roots of unity, show that

$$\sum_{j=0}^{n=1}|z_1+\omega^j z_2|^2 = n(|z_1|^2+|z_2|^2)$$

for any two complex numbers z_1 and z_2.

Let $\omega = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ and $\alpha = \omega + \omega^2 + \omega^4$ and $\beta = \omega^3 + \omega^5 + \omega^6$.

81. $\alpha + \beta$ equals
(a) 0 (b) –1 (c) –2 (d) 1

82. $\alpha\beta$ equals
(a) –1 (b) 0 (c) 1 (d) 2

83. α and β are roots of the equations
(a) $x^2 + x + 1 = 0$ (b) $x^2 + x + 2 = 0$
(c) $x^2 + 3x + 5 = 0$ (d) none of these

84. 2α equals
(a) $-1 + \sqrt{7}i$ (b) $-1 - \sqrt{7}i$
(c) $1 + 7i$ (d) $1 - 7i$

85. $\sum_{k=0}^{6} \omega^{k^2}$ equals
(a) i (b) $\sqrt{7}i$ (c) $-i$ (d) $-\sqrt{7}i$

Example 73 Value of

$$2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) \text{ is}$$

(a) n (b) -1

(c) $n-1$ (d) $n/3$

Example 74 If $n = 5$ in (1), then value of

$$(z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 + 2z\cos\frac{\pi}{5} + 1\right) \text{ is}$$

(a) $z^5 - 1$ (b) z^5

(c) $z^5 + 1$ (d) 0

Example 75 Value of $\sin\frac{\pi}{10}\cos\frac{\pi}{5}$ is

Paragraph for Example Nos. 71 to 75.

If $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are root nth roots of unity, then

$$\alpha_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

where $0 \le k \le n-1$. Also

$$x^n - 1 = (x - \alpha_0)(x - \alpha_1) \dots (x - \alpha_{n-1}) \qquad (1)$$

Example 71 Value of $(1 + \alpha_0)(1 + \alpha_1) \dots (1 + \alpha_{n-1})$ is

(a) 3 (b) $(-1)^n$

(c) 0 (d) $1 + (-1)^{n-1}$

Example 72 Value of $(1 - \alpha_1) \dots (1 - \alpha_{n-1})$ is

(a) n (b) $n-1$

(c) $(-1)^n$ (d) 0

Example 46 If $S = \sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$ then

(a) $S + \bar{S} = 0$ (b) $S\bar{S} = 1$

(c) $\sqrt{S} = \frac{1}{\sqrt{2}}(1 \pm i)$ (d) $S - \bar{S} = 0$

Ans. (a), (b), (c)

Example 30 Roots of the equations are

$$(z + 1)^5 = (z - 1)^5$$

are

(a) $\pm i \tan\left(\frac{\pi}{5}\right), \pm i \tan\left(\frac{2\pi}{5}\right)$

(b) $\pm i \cot\left(\frac{\pi}{5}\right), \pm i \cot\left(\frac{2\pi}{5}\right)$

(c) $\pm i \cot\left(\frac{\pi}{5}\right), \pm i \tan\left(\frac{2\pi}{5}\right)$

(d) none of these

Ans. (b)

If ω be a complex n^{th} root of unity, then $\sum_{r=1}^{n} (ar + b)\,\omega^{r-1}$ is equal to :

(A) $\frac{n(n+1)}{2}a$ (B) $\frac{nb}{1-n}$ (C*) $\frac{na}{\omega - 1}$ (D) none of these

Resolve $z^7 - 1$ into linear and quadratic factors and hence deduce that

$$\cos\frac{\pi}{7}\,.\,\cos\frac{2\pi}{7}\,.\,\cos\frac{4\pi}{7} = \frac{1}{8}.$$

Solve the equation $z^{10} - 1 = 0$ and deduce that,

$$\sin 5\theta = 5\sin\theta\left[1 - \frac{\sin^2\theta}{\sin^2(\pi/5)}\right]\left[1 - \frac{\sin^2\theta}{\sin^2(2\pi/5)}\right].$$

Let z be a root of $x^5 - 1 = 0$ with $z \neq 1$. Then the value of $z^{15} + z^{16} + z^{17} + \ldots + z^{50} =$

(A*) 1 (B) –1 (C) 0 (D) 5

If 1, $z_1, z_2, \ldots, z_{10}$ are the 11th roots of unity then match the entries given in column A with one of the entries given in column B.

	Column A		Column B
(A)	$(1 - z_1)(1 - z_2)\ldots(1 - z_{10}) =$	(1)	1
(B)	$1 + z_1^{100} + z_2^{100} + \ldots + z_{10}^{100} =$	(2)	– 1
(C)	$(1 + z_1)(1 + z_2)\ldots(1 + z_{10}) =$	(3)	0
(D)	$z_1 . z_2 . z_3 \ldots z_{10} =$	(4)	11

[Ans.: A – 4; B – 3; C – 1; D – 1]

If $1, z_1, z_2, z_3, \ldots\ldots, z_{n-1}$ be the n^{th} roots of unity and ω be a non-real complex cube root of unity, then the product $\prod_{r=1}^{n-1}(\omega - z_r)$ can be equal to :

(A) 0 (B) 1 (C) -1 (D*) $1+\omega$

The value of $\sum_{n=1}^{10}\left(\sin\frac{2n\pi}{11} - i\cos\frac{2n\pi}{11}\right)$ is :

(A*) i (B) $-i$ (C) 0 (D) 10

If Z_r, $r = 1, 2, 3, \ldots\ldots\ldots 2m$, $m \varepsilon N$ are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots\ldots\ldots\ldots + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m}\frac{1}{Z_r - 1} = -m$

Find the roots of the equation $Z^n = (Z+1)^n$ and show that the points which represent them are collinear on the complex plane. Hence show that these roots are also the roots of the equation

$$\left(2\sin\frac{m\pi}{n}\right)^2\bar{Z}^2 + \left(2\sin\frac{m\pi}{n}\right)^2\bar{Z} + 1 = 0.$$

If the equation $(z+1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots. z_7$, find the value of

(a) $\sum_{r=1}^{7}\text{Re}(Z_r)$ and (b) $\sum_{r=1}^{7}\text{Im}(Z_r)$

If ω is the fifth root of 2 and $x = \omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.

If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z-2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

Resolve $Z^5 + 1$ into linear & quadratic factors with real coefficients. Deduce that : $4\cdot\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{5} = 1$.

18. If $1, a_1, a_2, \ldots, a_{n-1}$ are the n roots of unity, then show that $(1-a_1)(1-a_2)(1-a_3)\ldots(1-a_{n-1}) = n$ **(1984, 2M)**

Match the Columns

13. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right); k = 1, 2, \ldots 9.$

	Column I		Column II
P.	For each z_k, there exists a z_j such that $z_k \cdot z_j = 1$	(i)	True
Q.	There exists a $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers	(ii)	False
R.	$\frac{\|1-z_1\|\|1-z_2\|\ldots\|1-z_9\|}{10}$ equal	(iii)	1
S.	$1 - \sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	(iv)	2

(2011)

8. Let z_1 and z_2 be nth roots of unity which subtend a right angled at the origin, then n must be of the form (where, k is an integer)

(a) $4k+1$ (b) $4k+2$ (c) $4k+3$ (d) $4k$

12. The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is

(a) -1 (b) 0 (c) $-i$ (d) i

The minimum value of $|a + b\omega + c\omega^2|$, where a,b and c are all not equal integers and $\omega(\neq 1)$ is a cube root of unity, is

(a) $\sqrt{3}$ (b) 1/2 (c) 1 (d) 0

Q. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r,s) for which $P^2 = -I$ is

Q. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n =

(A) 57 (B) 55 (C) 58 (D) 56

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is – (A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

Paragraph for Q Nos 89 & 90

Let a, b, and c be three real numbers satisfying

$$[abc]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [000] \ldots (E)$$

89. If the point $P(a, b, c,)$, with referenc to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is **[IIT - 2011]**

(A) 0 (B) 12 (C) 7 (D) 6

90. Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of **[IIT - 2011]**

$$\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$$

is equal to

(A) –2 (B) 2 (C) 3 (D) –3

Example 47 If $|z_1| = |z_2| = |z_3| = 1$ and

$$S = |z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$$

Let α = minimum value of S

and β = maximum value of S

as z_1, z_2, z_3 vary on $|z| = 1$, then

(a) $\alpha = 0$ (b) $\beta = 9$

(c) $\beta = 3$ (d) $\alpha = 2$

Ans. (a), (b)

Example 35 If $|z_1| = 2, |z_2| = 3, |z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$, then absolute value of $8z_2z_3 + 27z_3z_1 + 64z_1z_2$ equals

(a) 24 (b) 48 (c) 72 (d) 96

Ans. (d)

Example 63 If $\omega \neq 1$ is a cube root of unity and ω and ω^2 satisfy the equation

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} = \frac{2}{x}$$

then value of $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1}$ is

(a) 1 (b) 2

(c) 3 (d) 0

Ans. (b)

If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer. Find N.

The equation $x^3 = 9 + 46i$ where $i = \sqrt{-1}$ has a solution of the form $a + bi$ where a and b are integers. Find the value of $(a^3 + b^3)$.

Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]\cdots\cdots\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$$ is equal to $\left(1 - \frac{1}{2^{2^n}}\right)(1+i)$ where $n \geq 2$

If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with $a, b, c > 0$; $2b^2 > 4ac > b^2$; $z_1 \in$ third quadrant ; $z_2 \in$ second quadrant in the argand's plane then, show that

$$\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$$

(d) the Y-axis for $a = 0, b \neq 0$

7. Let $a, b \in R$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{z \in C : z = \frac{1}{a + i\,bt}, t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on **(2016 Adv.)**

(a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(c) the X-axis for $a \neq 0, b = 0$

40. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.

If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ & if $\cot\theta = x + 1$, then $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta}$ is equal to :

(A*) $\frac{\sin n\theta}{\sin^n\theta}$ (B) $\frac{\cos n\theta}{\cos^n\theta}$ (C) $\frac{\sin n\theta}{\cos^n\theta}$ (D) $\frac{\cos n\theta}{\sin^n\theta}$

10. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az + b}{z + 1}\right) = y$, then which of the following is(are) possible value(s) of x? **(2017 Adv.)**

(a) $1 - \sqrt{1 + y^2}$ (b) $-1 - \sqrt{1 - y^2}$

(c) $1 + \sqrt{1 + y^2}$ (d) $-1 + \sqrt{1 - y^2}$

If $\cos\alpha + a\cos\beta + b\cos\gamma = 0 = \sin\alpha + a\sin\beta + b\sin\gamma$ then :

(A*) $\cos 3\alpha + a^3\cos 3\beta + b^3\cos 3\gamma = 3ab \,.\, \cos(\alpha + \beta + \gamma)$

(B*) $\sin 3\alpha + a^3\sin 3\beta + b^3\sin 3\gamma = 3ab \,.\, \sin(\alpha + \beta + \gamma)$

(C*) $e^{i\alpha} + a e^{i\beta} + b e^{i\gamma} = 0$

(D*) $e^{i(\alpha + \pi/2)} + a e^{i(\beta + \pi/2)} + b e^{i(\gamma + \pi/2)} = 0$

If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2$ then prove that :

(a) $\Sigma\cos 2\alpha = 0 = \Sigma\sin 2\alpha$ (b) $\Sigma\sin(\alpha + \beta) = 0 = \Sigma\cos(\alpha + \beta)$

(c) $\Sigma\sin^2\alpha = \Sigma\cos^2\alpha = 3/2$ (d) $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$

(e) $\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$

(f) $\cos^3(\theta + \alpha) + \cos^3(\theta + \beta) + \cos^3(\theta + \gamma) = 3\cos(\theta + \alpha) . \cos(\theta + \beta) . \cos(\theta + \gamma)$ where $\theta \in R$.

If $\arg(z^{1/3}) = \frac{1}{2}\arg\left(z^2 + \bar{z}\,z^{1/3}\right)$, find the value of $|z|$.

Dividing f(z) by $z - i$, we get the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

23. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in R, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE? **(2018 Adv.)**

(a) If L has exactly one element, then $|s| \neq |t|$

(b) If $|s| = |t|$, then L has infinitely many elements

(c) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(d) If L has more than one element, then L has infinitely many elements

21. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[2012]**

(a) either on the real axis or on a circle passing through the origin.

(b) on a circle with centre at the origin

(c) either on the real axis or on a circle not passing through the origin.

(d) on the imaginary axis.

4. Let $z = \cos\theta + i\sin\theta$. Then, the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is **(2009)**

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$

(c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

Find all real values of the parameter a for which the equation $(a-1)z^4 - 4z^2 + a + 2 = 0$ has only pure imaginary roots.

9. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then, a cannot take the value **(2012)**

(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the co-ordinate axes in the Argand's plane. Further show that $|z| = \sqrt{\frac{-\text{Im}(z)}{2\text{Re}(z)\text{Im}(z)+1}}$

(b) The function $f: R \to (-\pi, \pi]$, defined by $f(t) = \arg(-1+it)$ for all $t \in R$, is continuous at all points of R, where $i = \sqrt{-1}$.

(c) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π.

(d) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line.

8. For a non-zero complex number z, let $\arg(z)$ denote the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) FALSE ? **(2018 Adv.)**

(a) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

The solution set of the equation, $z^2 + (3+2i)z - 7 + 17i = 0$ where z is a complex number expressed in the form of a + bi is ______. **[Ans.: 2 – 3i; –5 + i] [Hint: D = $\sqrt{33-56i}$ = 7 – 4 i or –7 + 4i]**

1. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) < 0$, is _____

Ans. 8

A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane & the circumcentre of the Δ ABC lies at the origin. If the altitude AD of the triangle meets the circumcircle again at P, then P represents the complex number:

(A) $-\frac{z_1 z_2}{z_3}$ (B*) $-\frac{z_2 z_3}{z_1}$ (C) $-\frac{z_3 z_1}{z_2}$ (D) none of these

The vector $z = -4 + 5i$ is turned counter clockwise through an angle of 180° & stretched 1.5 times . The complex number corresponding to the newly obtained vector is :

(A*) $6 - \frac{15}{2}i$ (B) $-6 + \frac{15}{2}i$ (C) $6 + \frac{15}{2}i$ (D) none of these

The points A, B, C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that

$$(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}.$$

Let z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that :

(i) $2z_2 = (1+i)z_1 + (1-i)z_3$ & (ii) $2z_4 = (1-i)z_1 + (1+i)z_3$

If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p , q , r are the moduli of non−zero complex numbers u, v, w respectively,

prove that, $\arg\frac{w}{v} = \arg\left(\frac{w-u}{v-u}\right)^2$.

P is a point on the Aragand diagram. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin & P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2 . \cos 2\theta = Z_1 . Z_3 \cos^2\theta$.

14. Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$.
(1997, 5M)

10. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1+i$ and $2-i$ respectively, then A represents the complex number ...or... **(1993, 2M)**

11. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i, z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a = \ldots$ and $b = \ldots$. **(1990, 2M)**

8. Let $W = \frac{\sqrt{3}+i}{2}$ and $P = \{W^n : n = 1,2,3,\ldots\}$.

Further $H_1 = \left\{z \in C : \text{Re}(z) > \frac{1}{2}\right\}$

and $H_2 = \left[z \in C : \text{Re}(z) < \frac{-1}{2}\right]$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2$ is equal to
(2013 JEE Adv.)

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{6}$

(c) $\frac{2\pi}{3}$

(d) $\frac{5\pi}{6}$

6. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is
(2001, 1M)

(a) of area zero

(b) right angled isosceles

(c) equilateral

(d) obtuse angled isosceles

5. If $0 < \alpha < \frac{\pi}{2}$ is a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = \{\cos(\alpha - \theta), \sin(\alpha - \theta)\}$, then Q is obtained from P by (2002, 2M)

(a) clockwise rotation around origin through an angle α
(b) anti-clockwise rotation around origin through an angle α
(c) reflection in the line through origin with slope $\tan\alpha$
(d) reflection in the line through origin with slope $\tan\frac{\alpha}{2}$

2. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anti-clockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by (2008, 3M)

(a) $6 + 7i$ (b) $-7 + 6i$ (c) $7 + 6i$ (d) $-6 + 7i$

3. A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then, the position of P in the Argand plane is (2007, 3M)

(a) $3e^{i\pi/4} + 4i$ (b) $(3-4i)e^{i\pi/4}$ (c) $(4+3i)e^{i\pi/4}$ (d) $(3+4i)e^{i\pi/4}$

18. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then, prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if; $\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

Example Let $\bar{a}z + a\bar{z} = c$, $a \neq 0$ be a line in the complex plane. If a point z_1 is the reflection of a point z_2 is the reflection of a point z_2 in the given line, then show that $c = \bar{z}_1 a + z_2 \bar{a}$

Reflection points for a straight line

Two given points P & Q are the reflection points for a given straight line if the given line is the right

Length of perpendicular

Length of perpendicular from ω to the line

$\bar{a}z + a\bar{z} + C = 0$ (where C is real)

is $|\bar{a}\omega + a\bar{\omega} + C| / 2|a|$

Proof :

Taking $a = \alpha + i\beta$, $z = x + iy$, the given line becomes $2(\alpha x + \beta y) + c = 0$.

Length of perpendicular from $\omega = x_1 + iy_1$ to the given line is

$$\frac{|2(\alpha x_1 + \beta y_1) + C|}{2\sqrt{\alpha^2 + \beta^2}} = \frac{|a\bar{\omega} + \bar{a}\omega + C|}{2|a|}$$

Let z_1, z_2, z_3 be three complex numbers and a, b, c be real numbers not all zero, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$. Show that z_1, z_2, z_3 are collinear.

A function f is defined on the complex number by $f(z) = (a + bi)z$, where 'a' and 'b' are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = \frac{u}{v}$ where u and v are coprimes. Find the value of (u + v).

(a) $\mu = \mu'$ (b) $\mu\mu' = -1$

(c) $\mu\mu' = 1$ (d) $\mu = \mu' e^{i\pi/2}$

38. If a line makes an angle $\theta (\neq \pi/2)$ with the positive direction of the x-axis, then its complex slope equals

(a) $e^{2i\theta}$ (b) $e^{-2i\theta}$

(c) $ie^{2i\theta}$ (d) none of these

Paragraph for Question Nos. 35 to 38

If z_1, z_2 are two complex numbers representing points A and B, we define complex slope of line AB as $\mu = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

35. Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ where $a \in \mathbf{C}$ and $b \in \mathbf{R}$ is

(a) $\dfrac{a}{\bar{a}}$ (b) $\dfrac{\bar{a}}{a}$

(c) $-\dfrac{a}{\bar{a}}$ (d) $-\dfrac{\bar{a}}{a}$

36. If μ and μ' are complex slopes of two perpendicular lines, then

(a) $\mu\mu' = -1$ (b) $\mu = \mu' e^{\pi i/2}$

(c) $\mu + \mu' = 0$ (d) $\mu\mu' = 1$

37. If μ and μ' are complex slopes of two parallel lines then

Example 50 If z_1, z_2, z_3, z_4 are the vertices of a square in that order, then

(a) $z_1 + z_3 = z_2 + z_4$

(b) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_4| = |z_4 - z_1|$

(c) $|z_1 - z_3| = |z_2 - z_4|$

(d) $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

Ans. (a), (b), (c), (d)

Example 48 Suppose $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1| = |z_2| = |z_3| = |z_4| = 1$. If $z_1, z_2, z_3\ z_4$ are the vertices of a quadrilateral, then the quadrilateral must be a

(a) parallelogram (b) rhombus

(c) rectangle (d) square

Ans. (a), (c)

Example 18 Reflection of the line $\bar{a}z + a\bar{z} = 0$ in the real axis is

(a) $\bar{a}\bar{z} + az = 0$ (b) $\frac{\bar{a}}{a} = \frac{\bar{z}}{z}$

(c) $(a + \bar{a})(z + \bar{z}) = 0$ (d) none of these

Ans. **(a)**

The locus represented by the equation, $|z - 1| + |z + 1| = 2$ is :

(A) an ellipse with focii $(1, 0)$; $(-1, 0)$

(B) one of the family of circles passing through the points of intersection of the circles $|z - 1| = 1$ and $|z + 1| = 1$

(C) the radical axis of the circles $|z - 1| = 1$ and $|z + 1| = 1$

(D*) the portion of the real axis between the points $(1, 0)$; $(-1, 0)$ including both .

A and B represent z_1 and z_2 in the Argands plane. The complex slope of AB is defined to be $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

Prove that the two lines in the Argand's plane with complex slopes ω_1 and ω_2 will be perpendicular if and only if $\omega_1 + \omega_2 = 0$. Also find the condition for two lines with complex slopes ω_1 and ω_2 to be parallel.

If $a\,\text{CiS}\,\alpha$, $b\,\text{CiS}\,\beta$, $c\,\text{CiS}\,\gamma$ represent three distinct collinear points in an Argand's plane, then prove the following :

(i) $\Sigma\, ab \sin(\alpha - \beta) = 0$.

(ii) $(a\,\text{CiS}\,\alpha)\sqrt{b^2 + c^2 - 2bc\cos(\beta - \gamma)} \pm (b\,\text{CiS}\,\beta)\sqrt{a^2 + c^2 - 2ac\cos(\alpha - \gamma)}$

$\mp (c\,\text{CiS}\,\gamma)\sqrt{a^2 + b^2 - 2ab\cos(\alpha - \beta)} = 0$.

13. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of the point z_2 through the line, then show that $c = \bar{z}_1 b + z_2\bar{b}$. **(1997C, 5M)**

18. The points z_1, z_2, z_3 and z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$ **(1983, 1M)**

(c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

9. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If arg (w) denotes the principal argument of a non-zero complex number w, then **(2010)**

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (b) $\arg(z - z_1) = \arg(z - z_2)$

(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ (d) $\arg(z - z_1) = \arg(z_2 - z_1)$

Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundry.

Theorem 9. *Let ABC be a triangle center, and assume that the circumcircle of ABC coincides with the unit circle of the complex plane. Then the circumcenter, centroid, and orthocenter of ABC are given by* 0, $\frac{1}{3}(a + b + c)$, $a + b + c$, *respectively.*

Let m be a line in the complex plane defined by

$$(1 - i)z + (1 + i)\bar{z} = 4.$$

Let $z_1 = 2 + 2i$ be a point in the complex plane.

If the reflection of z_1 in m is z_2, then compute the value of

$$\overline{z_1}(1 + i) + z_2(1 - i).$$

Find the equation of the circle in complex form which touches the line $iz + \overline{z} + 1 + i = 0$ and the lines $(2-i)z = (2+i)\overline{z}$ and $(2+i)z + (i-2)\overline{z} - 4i = 0$ are the normals of the circle.

A $|z - (1 + \frac{i}{2})| = \frac{3}{\sqrt{2}}$

B $|z - (1 - \frac{i}{2})| = \frac{3}{2}$

C $|z - (1 + \frac{i}{2})| = \frac{3}{2\sqrt{2}}$

D $|z - (1 - \frac{i}{2})| = \frac{3}{2\sqrt{2}}$

The mirror image of the curve given by $\arg\left(\frac{z+i}{z-1}\right) = \frac{\pi}{4}$ in the line x - y = 0 is

A) $\arg\left(\frac{z+i}{z+1}\right) = \frac{\pi}{4}$

B) $\arg\left(\frac{z+1}{z-i}\right) = \frac{\pi}{4}$

C) $\arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{4}$

D) $\arg\left(\frac{z+i}{z-1}\right) = \frac{\pi}{4}$

Inverse points w.r.t. a circle

Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :

(i) the point O, P, Q are collinear and on the same side of O.

(ii) $OP \cdot OQ = \rho^2$.

Example C is the centre of a given circle $|z - c| = r$. A is a point represented by the complex number a inside the circle and B is a point represented by the complex number b outside the circle, such that C, A, B lie in a straight line and $CA \cdot CB = r^2$. Show that $b = c + \dfrac{r^2}{(\bar{a} - \bar{c})}$

Example 103 If the complex number z is such that $|z - 1| \le 1$ and $|z - 2| = 1$ find the maximum possible value $|z|^2$.

Ans. 3

Example 40 z_1 and z_2 lie on a circle with centre at the origin. The point of intersection z_3 of the tangents at z_1 and z_2 is given by

(a) $\frac{1}{2}(\bar{z}_1 + \bar{z}_2)$ (b) $\frac{2z_1 z_2}{z_1 + z_2}$

(c) $\frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$ (d) $\frac{z_1 + z_2}{\bar{z}_1 \bar{z}_2}$

Ans. (b)

Example 32 Let z_1, z_2, z_3 be three distinct complex numbers lying on a circle with centre at the origin such that $z_1 + z_2 z_3$, $z_2 + z_3 z_1$ and $z_3 + z_1 z_2$ are real numbers, then $z_1 z_2 z_3$ equals

(a) -1 (b) 0

(c) 1 (d) none of these

Ans. (c)

Consider two complex numbers, $Z_1 = -3 + 2i$ & $Z_2 = 2 - 3i$, Z is a complex number such that, $\arg\left(\frac{Z - Z_1}{Z_2 - Z}\right) = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$ then, the locus of Z is :

(A) the minor arc of the circle with centre $(-4, -4)$ and radius $\sqrt{63}$

(B) an arc of the circle with centre $\left(-\frac{4}{3}, -\frac{4}{3}\right)$ and radius $\frac{5\sqrt{5}}{3}$

(C) the circle with centre $\left(-\frac{8}{3}, -\frac{8}{3}\right)$ and radius $\frac{\sqrt{221}}{3}$

(D) the circle with centre $\left(-\frac{8}{6}, -\frac{8}{6}\right)$ and radius $\frac{5\sqrt{5}}{3}$

If z lies on the circle centered at origin. If area of the triangle whose vertices are z, ωz and $z + \omega z$, where ω is the cube root of unity, is $4\sqrt{3}$ sq. units. Then radius of the circle is

(A) 1 unit (B) 2 units (C) 3 units (D*) 4 units

If the complex number z satisfies the equation $|z+1|^2 + |z-1|^2 = 4$, then the point z lies on:

(A) a straight line (B*) a circle (C) a parabola (D) none

Consider a square OABC, where O is origin, $A(z_0)$ and vertices of the square are inscribed in anticlockwise order. The equation of circle circumscribing the square is:

(A*) $\left|z - \frac{z_0(1+i)}{2}\right| = \frac{|z_0|}{\sqrt{2}}$ (B) $\left|z - \frac{z_0(1+i)}{2}\right| = \frac{|z_0|}{2}$

(C) $|z - z_0(1+i)| = |z_0|$ (D*) $\text{Arg}\left(\frac{z - z_0}{z - z_0 i}\right) = \pm\frac{\pi}{2}$

Let z be a complex number having the argument θ, $0 < \theta < \pi/2$ and satisfying the equality $|z - 3i| = 3$. Then $\cot\theta - \frac{6}{z}$ is equal to:

(A) 1 (B) -1 (C*) i (D) $-i$

Let S denote the set of all complex numbers z satisfying the inequality $|z - 5i| \le 3$. Find the complex numbers z in S having (i) least positive argument (ii) maximum positive argument (iii) least modulus and (iv) maximum modulus.

Among the complex numbers 'z', find 'z' which satisfies $|z - 25i| \le 15$ and having least positive argument ________

If there is atleast one complex number z which satisfies both the equality ; $|z - mi| = m + 5$ & the inequality $|z - 4| < 3$, then the real values of the parameter m are given by :

(A) $-1 < m < 1$ (B) $-2 < m < 2$ (C*) $-3 < m < 3$ (D) none

The real values of the parameter 'a' for which atleast one complex number $z = x + iy$ $(x, y \in R)$ satisfies the equality $|z + \sqrt{2}| = a^2 - 3a + 2$ & the inequality $|z + i\sqrt{2}| < a^2$ is ______ **Ans. : a > 2**

If $|z + 4| \le 3$, then the maximum value of $|z + 1|$ is

[1] 6 [2] 0 [3] 4 [4] 10

If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[IIT - 2011]**

ANSWER : 5

C is the complex number. $f : C \to R$ is defined by $f(z) = |z^3 - z + 2|$. What is the maximum value of f on the unit circle $|z| = 1$?

If the complex number P(w) lies on the standard unit circle in an Argand's plane and $z = (aw + b)(w - c)^{-1}$ then, find the locus of z and interpret it. Given a, b, c are real.

12. If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2+\sqrt{3}i$. Find the other vertices of square. **(2005, 4M)**

48. If z is any complex number satisfying $|z-3-2i| \le 2$, then the minimum value of $|2z-6+5i|$ is ...... **(2011)**

26. $\min_{z \in s} |1-3i-z|$ is equal to

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$

(c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

27. Area of S is equal to

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$ (c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

Passage II

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in C : |z| < 4\}, S_2 = \left\{z \in C : \text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$

and $S_3 : \{z \in C : \text{Re } z > 0\}$ **(2008)**

Paragraph for Question Nos. 21 to 23

Let A, B, C be three sets of complex numbers as defined below

$A = \{z : \text{Im} z \ge 1\}$

$B = \{z : |z-2-i| = 3\}$

$C = \{z : \text{Re}((1-i)z) = \sqrt{2}\}$.

21. The number of elements in the set $A \cap B \cap C$ is

(A) 0 (B) 1 *Sol.* **(B)**

(C) 2 (D) ∞

22. Let z be any point in $A \cap B \cap C$. Then, $|z+1-i|^2 + |z-5-i|^2$ lies between

(A) 25 and 29 (B) 30 and 34 *Sol.* **(C)**

(C) 35 and 39 (D) 40 and 44

23. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w-2-i| < 3$. Then, $|z| - |w| + 3$ lies between

(A) -6 and 3 (B) -3 and 6

(C) -6 and 6 (D) -3 and 9 *Sol.* **(D)**

8. Let complex numbers α and $1/\bar{\alpha}$ lies on circles $(x-x_0)^2 + (y-y_0)^2 = r^2$ and $(x-x_0)^2 + (y-y_0)^2 = 4r^2$, respectively.
If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha|$ is equal to **(2013 Adv.)**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

3 Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2 - 3 - 4i| = 4$. Then, the minimum value of $|z_1 - z_2|$ is **(2019 Main, 12 Jan II)**

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) 0

7. If z is a complex number such that $|z| \ge 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$

(a) is equal to 5/2

(b) lies in the interval (1, 2)

(c) is strictly greater than 5/2

(d) is strictly greater than 3/2 but less than 5/2

The complex number z satisfying $|z+2+i| + |z-2+i| = 4$, $0 \le \arg(z+2+2i) \le \frac{\pi}{4}$ $3\frac{\pi}{4} \le \arg(z-2+2i) \le \pi$ will lie on a line segment of the length k. Find k.

If r_1 and r_2 are the maximum and minimum distances of a points on the curve $10(z\bar{z}) - 3i\{z^2 - (\bar{z})^2\} - 6 = 0$ from origin then find $r_1 + r_2$.

42. Let eccentricity of ellipse with focii z_1 and z_2 be 1/3. If $|z_1 - z_2| = 5$ and z lies on the ellipse, find maximum possible value of $|z - z_1|$

Example 112 Match the equation on the left with the curve they represent on the right

(a) $|z - 3| + |z - i| = 10$	(p) circle
(b) $\left\lvert\dfrac{2z-3}{z-i}\right\rvert = 2$	(q) hyperbola
(c) $z^2 + \bar{z}^2 = 5$	(r) straight line
(d) $\left\lvert\dfrac{z-6}{z-2i}\right\rvert = 3$	(s) ellipse

Ans.

	p	q	r	s
a	p	q	r	**s**
b	p	q	**r**	s
c	p	**q**	r	s
d	**p**	q	r	s

Example 15 If $a > 0$, and the equation $|z - a^2| + |z - 2a| = 3$ represents an ellipse, then a lies in

(a) (1, 3) (b) $(\sqrt{2}, \sqrt{3})$

(c) (0, 3) (d) $(1, \sqrt{3})$

Ans. (c)

If 'z' be a complex number on the complex plane satisfying, $|z| - |z - 6 - 8i| = 10$, then 'z' represents :

(A) a hyperbola (B*) a line segment (C) a circle (D) a parabola

- **Ellipse:** $|z-z_1|+|z-z_2|=k$ represents

i) an ellipse if $k>|z_1-z_2|$ with z_1, z_2 as its foci, k is the length of major axis

ii) an empty set if $k<|z_1-z_2|$

iii) a line segment if $k=|z_1-z_2|$

- **Hyperbola:** $||z-z_1|-|z-z_2||=k$ represents

i) a hyperbola if $k<|z_1-z_2|$ with z_1, z_2 as its foci and k is length of transversal axis

ii) an empty set if $k>|z_1-z_2|$

iii) two rays if $k=|z_1-z_2|$

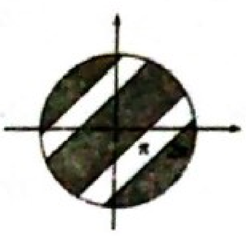
Integer answer type questions

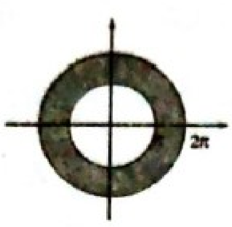
Let A, B, C three set of complex number defined below

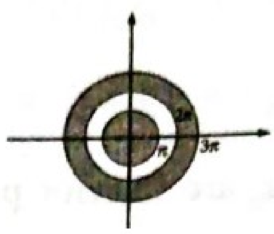
$A=\{z:|z+1|\le 2+\text{Re}(z)\}$; $B=\{z:|z-1|\ge 1\}$; $C\left\{z:\left|\dfrac{z-1}{z+1}\ge 1\right|\right\}$

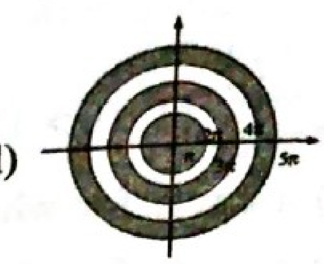
The number of points having integral coordinate in region $A\cap B\cap C$ is

The region described by the complex number z satisfying $\sin|z|>0$ is

a) 

b) 

c) 

d) 

The complex number where the curves arg (z − 3 i) = 3π/4 & arg (2z + 1 − 2 i) = −π/4 intersect is ______.

Let $z=x+iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A=\{z \mid |z|\le 2\}$ and $B=\{z \mid (1-i)z+(1+i)\bar{z}\ge 4\}$. Find the area of the region $A\cap B$.

Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ (b) $\text{Re}\left(\frac{z+2i}{iz+2}\right) \le 4 \quad (z \ne 2i)$

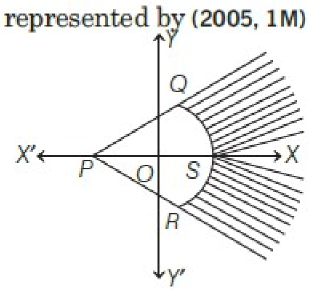
(c) $\text{Arg}(z + i) - \text{Arg}(z - i) = \pi/2$ (d) $\text{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$.

4. The shaded region, where $P = (-1, 0), Q = (-1 + \sqrt{2}, \sqrt{2})$ $R = (-1 + \sqrt{2}, -\sqrt{2}), S = (1, 0)$ is represented by **(2005, 1M)**

(a) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

(b) $|z + 1| < 2, |\arg(z + 1)| < \frac{\pi}{2}$

(c) $|z + 1| > 2, |\arg(z + 1)| > \frac{\pi}{4}$



Passage-1:

Let A(z_1), B(z_2), C(z_3) and D (z_4) be the vertices of a trapezium in an argand plane. Let $|z$

$|z_3 - z_4| = 10$ and the diagonals AC and BD intersect at P. it is given that arg $\left(\frac{Z_4 - Z_2}{Z_3 - Z_1}\right.$

$\arg\left(\frac{Z_3 - Z_2}{Z_4 - Z_1}\right) = \frac{\pi}{4}$.

39. Area of the trapezium ABCD is equal to

a) $\frac{130}{3}$ b) $\frac{160}{3}$ c) $\frac{190}{3}$ d) $\frac{140}{3}$

40. Area of triangle PCB is equal to

a) $\frac{100}{21}$ b) $\frac{200}{21}$ c) $\frac{100}{7}$ d) $\frac{400}{21}$

41. $|CP - DP|$ is equal to

a) $\frac{10}{\sqrt{21}}$ b) $\frac{16}{\sqrt{21}}$ c) $\frac{17}{\sqrt{21}}$ d) $\frac{19}{\sqrt{21}}$

Example 69 Suppose $A(z_1)$, $B(z_2)$ and $C(z_3)$ are vertices of a triangle lying on the unit circle $|z| = 1$. AD is altitude of the ΔABC meeting the unit circle in E.

(a) orthocentre of ΔABC is $z_1 + z_2 + z_3$

(b) affix of E is $-z_2 z_3/z_1$

(c) if $z_1^2 = z_2 z_3$ and $z_2^2 = z_3 z_1$, then ΔABC is equilateral.

(d) if $z_2 + z_3 = 0$, then ΔABC is a right angled.

Ans. (a), (b), (c), (d)

For all real numbers x, let the mapping $f(x) = \frac{1}{x - i}$, where $i = \sqrt{-1}$. If there exist real number a, b, c and d for which $f(a), f(b), f(c)$ and $f(d)$ form a square on the complex plane. Find the area of the square.

PASSAGE - IV

Consider a complex valued function $f(z)=\frac{1}{z-i}$

42. The locus of point $f(z)$ as 'z' lies on real axis is

A) a straight line B) a circle
C) an ellipse D) parabola

43. If $f(a), f(b), f(c)$ $\left(a, b, c \in R-\{0\}\right)$ are the vertices of a triangle then maximum area of triangle is

A) $\frac{3\sqrt{3}}{4}$ B) $\frac{1}{16}$ C) $\frac{3\sqrt{3}}{16}$ D) $\frac{1}{8}$

44. If $f(a), f(b), f(c), f(d)\left(a, b, c, d \in R-\{0\}\right)$ be the vertices of a square then area of the square

A) depends upon a, b, c, d
B) can't be determined
C) is always constant
D) independent on a, b, c

Let A_k (k = 1, 2, ... n) be the vertices of a regular n – gon inscribed in a unit circle then prove that, $\prod_{r=2}^{r=n}\left|A_1A_r\right| = n.$

Angle bisector

Given in the complex plane are two points z_1 and z_2. Let us find the complex numbers corresponding to the points lying on the bisector of the angle formed by the vector z_1 and z_2.

We know that the diagonal of a rhombus bisects the angle at the vertex. If we add two unit vectors then the resultant is along the angle bisector.

Thus, $\frac{z_1}{|z_1|}$ is an unit vector along z_1 and similarly

$\frac{z_2}{|z_2|}$ is a unit vector along z_2.

Their sum $\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}$ is a vector along the angle bisector.

A general vector along the angle bisector is

$t\left(\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right)$ where t is a positive real number.

Example 115 z_1, z_2, z_3 are vertices of a triangle. Match the condition on the left with type of triangle on the right

(a) $z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$ (p) right angled

(b) $Re\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = 0$ (q) obtuse angled

(c) $Re\left(\frac{z_3 - z_1}{z_3 - z_2}\right) < 0$ (r) isosceles and right angled

(d) $\frac{z_3 - z_1}{z_3 - z_2} = i$ (s) equilateral

Ans.

	p	q	r	s
a	p	q	r	**s**
b	**p**	q	r	s
c	p	**q**	r	s
d	p	q	**r**	s

Example 38 Let $r_k > 0$ and $z_k = r_k (\cos \alpha_k + i \sin \alpha_k)$ for $k = 1, 2, 3$ be such that

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$$

Let A_k be the point in the complex plane given by $w_k = \dfrac{\cos 2\alpha_k + i \sin 2\alpha_k}{z_k}$ for $k = 1, 2, 3$. The origin, O is the

(a) incentre of $\Delta A_1A_2A_3$
(b) orthocentre of $\Delta A_1A_2A_3$
(c) circumcentre of $\Delta A_1A_2A_3$
(d) centroid of $\Delta A_1A_2A_3$

Ans. (d)

Example 25 If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are z_1, z_2, z_3 is

(a) $3\sqrt{3}/4$ (b) $\sqrt{3}/4$ (c) 1 (d) 2

Ans. (a)

Example 16 If θ is real and z_1, z_2 are connected by $z_1^2 + z_2^2 + 2z_1 z_2 \cos \theta = 0$, then triangle with vertices 0, z_1 and z_2 is

(a) equilateral (b) right angled
(c) isosceles (d) none of these.

Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2}\,|z|^2$ CiS $\left(\frac{\pi}{4} + \theta\right)$ where θ = amp z . If 'O' is the origin, then the Δ OPQ is :

(A) isosceles but not right angled (B) right angled but not isosceles
(C*) right isosceles (D) equilateral

Prove that the roots of the equation $\frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} = 0$ where z_1, z_2, z_3 are pairwise distinct complex numbers, correspond to points on a complex plane which lie inside a triangle with vertices z_1, z_2, z_3 or on its sides .

Let z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 is:
(A*) an isosceles right angled triangle
(B) a right angled triangle which is not isosceles
(C) an equilateral triangle
(D) an isosceles triangle which is not right angled.

If the roots of the equation $Z^3 + 3aZ^2 + 3bZ + c = 0$ correspond to the vertices of an equilateral triangle, prove that $a^2 = b$.

Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$. If $z_1 + z_2 + z_3 = 3$ then z_1, z_2, z_3 must represent the vertices of:
(A*) an equilateral triangle
(B) an isosceles triangle which is not equilateral
(C) a right triangle
(D) nothing definite can be said.

Let z_1, z_2 & z_3 be the complex numbers representing the vertices of a triangle ABC respectively. If P is a point representing the complex number z_0 satisfying;
$a(z_1 - z_0) + b(z_2 - z_0) + c(z_3 - z_0) = 0$ then w.r.t. the triangle ABC, the point P is its:
(A) centroid (B) orthocentre (C) circumcentre (D*) incentre

If the area of the triangle on the complex plane formed by the points z, z + iz & iz is 50 then |z| is:

(A) 1 (B) 5 (C*) 10 (D) 15

Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then **[AIEEE- 2003]**

[1] $a^2 = b$ [2] $a^2 = 2b$ [3] $a^2 = 3b$ [4] $a^2 = 4b$

Let $A \equiv z_1$; $B \equiv z_2$; $C \equiv z_3$ are three complex numbers denoting the vertices of an acute angled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that

$$z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$$

hence show that the Δ ABC is a right angled triangle $\Leftrightarrow z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1 = 0$

17. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$. **(1983, 2M)**

15. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

(c) Incentre of the Δ ABC is

$$(az_1 + bz_2 + cz_3) \div (a + b + c).$$

(d) Circumcentre of the Δ ABC is

$$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C).$$

(ii) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(a) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$

(b) $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

(i) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively, then :

(a) Centroid of the Δ ABC is $\dfrac{z_1+z_2+z_3}{3}$

(b) Orthocentre of the Δ ABC is

$$\frac{(a\sec A)z_1+(b\sec B)z_2+(c\sec C)z_3}{a\sec A+b\sec B+c\sec C}$$

or $$\frac{z_1\tan A+z_2\tan B+z_3\tan C}{\tan A+\tan B+\tan C}$$

9. Let S be the set of all complex numbers z satisfying $|z^2+z+1|=1$. Then which of the following statements is/are TRUE ?

(A) $\left|z+\frac{1}{2}\right|\le\frac{1}{2}$ for all $z\in S$

(B) $|z|\le 2$ for all $z\in S$

(C) $\left|z+\frac{1}{2}\right|\ge\frac{1}{2}$ for all $z\in S$

(D) The set S has exactly four elements

Ans. (B,C)