Let $f(x)$ and $g(x)$ be two differentiable functions on R (the set of all real numbers) satisfying $f(x) = \frac{x^3}{2} + 1 - x\int_0^x g(t)\,dt$ and $g(x) = x - \int_0^1 f(t)\,dt$.

31. The value of definite integral $\int_0^1 f(t)\,dt$ lies in the interval :

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 1\right)$ (c) $\left(1, \frac{4}{3}\right)$ (d) $\left(\frac{4}{3}, \frac{5}{3}\right)$

32. Minimum vertical distance between the two curves $f(x)$ and $g(x)$ is :

(a) $\frac{7}{3}$ (b) $\frac{1}{6}$ (c) $\frac{8}{3}$ (d) $\frac{7}{6}$

33. If the distance of the point $P(x_1, y_1)$ on the curve $y = f(x)$ from the curve $y = g(x)$ is least, then x_1 equals :

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{7}{6}$

34. Number of points where $f(|x|)$ is non-derivable, is :

(a) 0 (b) 1 (c) 2 (d) 3

Let $f(x) = x^2 + 2x - t^2$ and $f(x) = 0$ has two roots $\alpha(t)$ and $\beta(t)(\alpha < \beta)$ where t is a rea parameter. Let $I(t) = \int_\alpha^\beta f(x)\,dx$. If the maximum value of $I(t)$ be λ and $|\lambda| = \frac{p}{q}$ where p and q ar relatively prime positive integers. Find the product (pq).

Let $f(x) = \begin{cases} \left|x^3 + x^2 + 3x + \sin x\right|\left(3 + \sin\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then the number of points where $f(x)$ attains it minimum value is ________

Passage - IV :

Let $f'(\sin x) < 0$ *and* $f''(\sin x) > 0\ \forall\ x \in \left(0, \frac{\pi}{2}\right)$ *and* $g(x) = f(\sin x) + f(\cos x)$.

26. Which of the following is true in $\left(0, \frac{\pi}{2}\right)$?

a) g' is increasing
b) g' is decreasing
c) g' has a point of minima
d) g' has a point of maxima

27. Which of the following is true ?

a) $g(x)$ is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
b) $g(x)$ increasing in $\left(0, \frac{\pi}{4}\right)$
c) $g(x)$ is monotonically increasing
d) None of these

$$f(x)=x^2-4|x| \text{ and } g(x)=\begin{cases}\min\{f(t):-6\le t\le x\}, x\in[-6,0]\\ \max\{f(t):0<t\le x\}, x\in(0,6]\end{cases}$$

17. $g(x)$ is strictly increasing in

a) [4, 6] b) [–4, 4] c) [–6, 6] d) [–6, –4]

18. Minimum value of $g(x)$ is

a) 0 b) –4 c) 2 d) 12

19. $g(x)$ is constant in

a) (–2, 0) b) (–4, 4) c) (0, 6) d) (0, 4)

18. Let $f(x)=\log(2x-x^2)+\sin\frac{\pi x}{2}$. Then which of the following is/are true ?

a) graph of f is symmetrical about the line $x=1$ b) maximum value of f is 1

c) absolute minimum value of f does not exist b) $f(x)$ is a periodic function

The least value of "a" for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$ *for atleast one solution of* interval $(0,\pi/2)$ is

a) 1 b) 4 c) 8 d) 9

Example 47. A cylinder is obtained by revolving a rectangle about the x–axis, the base of the rectangle lying on the x–axis and the entire rectangle lying in the region between the curve $y=\frac{x}{x^2+1}$ and the x–axis. Find the maximum possible volume of the cylinder.

PASSAGE - 1

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

1. The line $y = x$ meets $y = ke^x$ for $k \le 0$ at *(2007 - 4 marks)*

(a) no point (b) one point

(c) two points (d) more than two points

2. The positive value of k for which $ke^x - x = 0$ has only one root is *(2007 - 4 marks)*

(c) e e (d) $\log_e 2$

3. For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is *(2007 - 4 marks)*

(a) $\left(0, \frac{1}{e}\right)$ (b) $\left(\frac{1}{e}, 1\right)$

(c) $\left(\frac{1}{e}, \infty\right)$ (d) $(0, 1)$

13. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0,\infty)$, then *(2012)*

(a) f has a local maximum at $x = 2$

(b) f is decreasing on $(2, 3)$

(c) there exists some $c \in (0, \infty)$, such that $f''(c) = 0$

(d) f has a local minimum at $x = 3$

69. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is .... (2009)

65. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is equal to (2012)

64. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is (2012)

51. Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that $f(x)$ has the smallest value at $x = 1$. (1993, 5M)

66. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is......

Passage Based Problems

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}; 0 < a < 2.$

39. Which of the following is true ?

(a) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(b) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(c) $f'(1) f'(-1) = (2 - a)^2$

(d) $f'(1) f'(-1) = -(2 + a)^2$

40. Which of the following is true ?

(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$

(b) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x = 1$

(c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$

(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$

41. Let $g(x) = \int_0^{e^x} \dfrac{f'(t)}{1 + t^2} dt$. Which of the following is true?

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(d) $g'(x)$ does not change sign $(-\infty, \infty)$

37. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$, then

(a) $f(x)$ is increasing on $[-1, 2]$

(b) $f(x)$ is continuous on $[-1, 3]$

(c) $f'(2)$ does not exist

(d) $f(x)$ has the maximum value at $x = 2$

35. If $f(x)$ is a cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18, f(1) = -1$ and $f'(x)$ has local minimum at $x = 0$, then **(2006, 3M)**

(a) the distance between $(-1, 2)$ and $(a, f(a))$, where $x = a$ is the point of local minima, is $2\sqrt{5}$

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$

(c) $f(x)$ has local minima at $x = 1$

(d) the value of $f(0) = 5$

20. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2 + x)^3, & -3 < x \le -1 \\ x^{\frac{2}{3}}, & -1 < x < 2 \end{cases}$ is **(2008, 3M)**

(a) 0 (b) 1 (c) 2 (d) 3

Q.18. Let the function $f:(0, \pi) \to \mathbb{R}$ be defined by

$$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4.$$

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, \ldots, \lambda_r\pi\}$, where $0 < \lambda_1 < \ldots < \lambda_r < 1$. Then the value of $\lambda_1 + \ldots + \lambda_r$ is _____

Sol. **0.5**

*13. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____

Sol. **8**

6. Consider all rectangles lying in the region

$$\left\{(x, y) \in R \times R : 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\dfrac{3\pi}{2}$ (B) π (C) $\dfrac{\pi}{2\sqrt{3}}$ (D) $\dfrac{\pi\sqrt{3}}{2}$

Sol. **C**

Q.5 Let $f(x)=\frac{\sin \pi x}{x^2}, x>0$

Let $x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f.

Then which of the following options is/are correct?

A. $x_1 < y_1$

B. $|x_n - y_n| > 1$ for every n

C. $x_n \in \left(2n, 2n+\frac{1}{2}\right)$ for every n

D. $x_{n+1} - x_n > 2$ for every n

Sol. **B, C, D**

Q. 6 Let $f: R \to R$ by given by $f(x)=\begin{cases} x^5+5x^4+10x^3+10x^2+3x+1, & x<0; \\ x^2-x+1, & 0\le x<1; \\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3}, & 1\le x<3; \\ (x-2)\log_e(x-2)-x+\frac{10}{3}, & x\ge 3. \end{cases}$

Then which of the following options is/are correct?

A. f′ has a local maximum at x = 1

B. f′ is NOT differentiable at x = 1

C. f is onto

D. f is increasing on $(-\infty, 0)$

Sol. **A, B, C**

Q. 1 Let $M=\begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$, where $\alpha=\alpha(\theta)$ and $\beta=\beta(\theta)$ are real numbers, and I is the 2×2 identity matrix. If α^* is the minimum of the set $\{\alpha(\theta): \theta \in [0,2\pi)\}$ and β^* is the minimum of the set $\{\beta(\theta): \theta \in [0,2\pi)\}$, then the value of $\alpha^* + \beta^*$ is

A. $-\frac{37}{16}$

B. $-\frac{31}{16}$

C. $-\frac{17}{16}$

D. $-\frac{29}{16}$

Sol. **D**

Q.48 If $f(x)=\begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

[A] $f'(x)=0$ at exactly three points in $(-\pi, \pi)$

[B] $f'(x)=0$ at more than three points in $(-\pi, \pi)$

[C] $f(x)$ attains its maximum at $x=0$

[D] $f(x)$ attains its minimum at $x=0$

Sol. **B, C**

17. Let $f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}$ then

a) $f(x)$ has global maximum

b) $f(x)$ has local minimum

c) $f(x)$ has absolute minimum

d) $f(x)$ has local maximum

45. Let $f:\mathbb{R}\to(0,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ be twice differentiable functions such that f'' and g'' are continuous functions on $\mathbb{R}$. Suppose $f'(2)=g(2)=0$, $f''(2)\neq 0$ and $g'(2)\neq 0$. If $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)}=1$, then

(A) f has a local minimum at x = 2

(B) f has a local maximum at x = 2

(C) $f''(2) > f(2)$

(D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

Sol. **(A, D)**

42. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$, then

(A) $f(x)$ has three real roots if $a > 4$ (B) $f(x)$ has only one real roots if $a > 4$

(C) $f(x)$ has three real roots if $a < -4$ (D) $f(x)$ has three real roots if $-4 < a < 4$

The lower corner of a leaf in a book is folded over so as to just reach the inner edge of the page. The fraction of width folded over if the area of the folded part is minimum is:

(A) 5/8 (B*) 2/3 (C) 3/4 (D) 4/5

Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5 & , x \le 1 \\ -2x + \log_2(b^2 - 2) & , x > 1 \end{cases}$ the set of values of b for which f(x) has greatest value at x = 1 is given by :

(A) $1 \le b \le 2$ (B) $b = \{1, 2\}$

(C) $b \in (-\infty, -1)$ (D*) $\left[-\sqrt{130}, -\sqrt{2} \right) \cup \left(\sqrt{2}, \sqrt{130} \right]$

Find the minimum value of $|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$ for all real x.

48. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x =

(A) -2 (B) $\frac{-2}{3}$

(C) 2 (D) $\frac{2}{3}$

Sol. **(A), (B)**

Q.10 Consider the function $f(x) = \begin{cases} \sqrt{x}\, \ln x & \text{when } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$

(a) Find whether f is continuous at x = 0 or not.

(b) Find the minima and maxima if they exist.

(c) Does $f'(0)$? Find $\lim_{x \to 0} f'(x)$.

(d) Find the inflection points of the graph of y = f (x)..

55. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are

(A) 24 (B) 32

(C) 45 (D) 60

Sol. **(A, C)**

For a > 0, find the minimum value of the integral $\int_0^{1/a} (a^3 + 4x - a^5x^2)e^{ax}\, dx$.

If $ax^2 + (b/x) \ge c$ for all positive x where a > 0 & b > 0 then show that $27ab^2 \ge 4c^3$.

Find the largest term in the sequence $\frac{1}{301}, \frac{8}{316}, \frac{27}{381}, \frac{64}{556}, \ldots$

Read the following write up carefully and answer the following questions:

Let $f(x) = 12x^2\int_0^1 yf(y)\,dy + 20\int_0^1 xy^2 f(y)\,dy + 4x$

16. The maximum value of f(x) is

(A) 8 (B) $\frac{1}{8}$

(C) 16 (D) $\frac{1}{16}$

17. The number of solutions of the equation $|f(|x|)| = e^{|x|}$

(A) 0 (B) 2

(C) 4 (D) 3

18. The range of $f(-2^x)$ is

(A) $(-\infty, 0)$ (B) $(0, \infty)$

(C) $\left(-\infty, \frac{1}{8}\right)$ (D) $\left(\frac{1}{8}, \infty\right)$

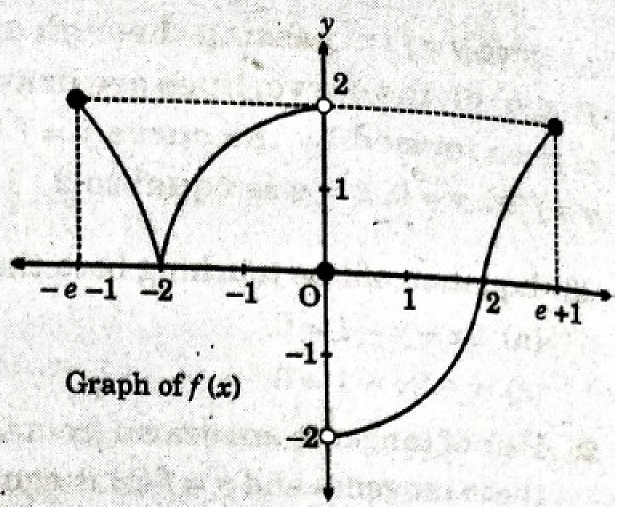
3. Let f be a continuous function and satisfies $f'(x) > 0$ on $(-\infty, \infty)$ and the value of $f''(x)\ \forall\ x \in (0, \infty)$ is equal to minimum value of $\min_{x\in R}\{e^{-|x|} + 2, |x| + 2\}$.

If $L = \lim_{x\to\infty} \dfrac{3x^2 - \dfrac{3}{x^2+1} - 4f'(x)}{f(x)}$ then find the value of $[L^2]$.

[Note : $[k]$ denotes greatest integer less than or equal to k.]

Paragraph for Questions Nos. 35 to 37

Let $f(x) = \begin{cases} 2\ln(-x-1), & -e-1 \le x \le -2 \\ \frac{1}{2}(4-x^2), & -2 < x < 0 \\ 0, & x = 0 \\ \frac{1}{2}(x^2-4), & 0 < x < 2 \\ 2\ln(x-1), & 2 \le x \le e+1 \end{cases}$



and graph of $f(x)$ is as shown.

Graph of $f(x)$

Also, $g(x) = \begin{cases} \min\{f(t) : -e-1 \le t \le x\}, & -e-1 \le x < 0 \\ \max\{f(t) : 0 \le t \le x\}, & 0 \le x \le e+1 \end{cases}$

35. Which one of the following statement does not hold good?

(a) Range of $g(x)$ is $[0, 2]$
(b) $g(x)$ is non-monotonic in $[-2, 2]$
(c) $g(x)$ is a continuous function
(d) $g(x)$ is an odd function

36. If $x = \alpha$ is the point of non-differentiability of $g(x)$ in $(-e-1, e+1)$ then the value of α is :

(a) $-2, 1$
(b) $-2, -1$
(c) $-2, 2$
(d) $-2, 0$

37. If the equation $g(x) = k$ has exactly two distinct solutions in $[-e-1, e+1]$ then the sum of all possible integral values of k is :

(a) 0
(b) 1
(c) 2
(d) 3

5. If $g(x)$ is twice differentiable real valued function satisfying $g''(x) - 3g'(x) > 3 \; \forall\, x$ ... $g'(0) = -1$, then $h(x) = g(x) + x \; \forall x > 0$ is :

(a) strictly increasing
(b) strictly decreasing
(c) non monotonic
(d) data insufficient

[2008]

66. Let the function $g: (-\infty, \infty) \to (-\pi/2, \pi/2)$ be given by $g(u) = 2\tan^{-1}(e^u) - \pi/2$, then g is

(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but in strictly increasing in $(-\infty, \infty)$. [2008]

Example 14 The function $g(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$ $(x \geq 0)$ is

(a) increasing on $[0, \infty)$

(b) decreasing on $[0, \infty)$

(c) increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$

(d) decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

Ans. (b)

Problem 9. Consider the function,

$f(x) = x^3 - 9x^2 + 15x + 6$ for $1 \leq x \leq 6$ and

$$g(x) = \begin{cases} \min. f(t) & \text{for } 1 \leq t \leq x,\ 1 \leq x \leq 6 \\ x - 18 & \text{for } x > 6 \end{cases}$$

then prove that

(i) g(x) is differentiable at x = 1 (ii) g(x) is discontinuous at x = 6 (iii) g(x) is continuous and derivable at x = 5

(iv) g(x) is monotonic in (1, 5)

Problem 7. Find the number of real roots of the equation $\displaystyle\sum_{i=1}^{n} \frac{a_i^2}{x - b_i} = c$, where $b_1 < b_2 < \ldots.. < b_n$.

Example 14. Find the intervals of increase of g(x), where $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall\ x \in R$, given that $f''(x) > 0\ \forall\ x \in R$.

Example 24. Find possible values of a such that $f(x) = e^{2x} - (a+1)e^x + 2x$ is strictly increasing for $x \in R$.

Q.7. Let b be a nonzero real number. Suppose f L $\mathbb{R} \to \mathbb{R}$ is a differentiable function such that f(0) = 1. If the derivative f′ of f satisfies the equation

$$f'(x) = \frac{f(x)}{b^2 + x^2}$$

for all $x \in \mathbb{R}$, then which of the following statements is/are TRUE?

(A) If $b > 0$, then f is an increasing function

(B) If $b < 0$, then f is a decreasing function

(C) $f(x)\, f(-x) = 1$ for all $x \in \mathbb{R}$

(D) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

Sol. **A, C**

17. For a polynomial g(x) with real coefficients, let m_g denote the number of distinct real roots of g(x). Suppose S in the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}.$$

For a polynomial f, let f′ and f″ denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Q.54 Which of the following options is the only **INCORRECT** combination ?

[A] (II) (iii) (P) [B] (II) (iv) (Q)

[C] (I) (iii) (P) [D] (III) (i) (R)

Sol. **D**

Column 1	**Column 2**	**Column 3**
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0, 1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

Q.52 Which of the following options is the only CORRECT combination ?

[A] (IV) (i) (S) [B] (I) (ii) (R)

[C] (III) (iv) (P) [D] (II) (iii) (S)

Sol. **D**

Q.53 Which of the following options is the only CORRECT combination ?

[A] (III) (iii) (R) [B] (I) (i) (P)

[C] (IV) (iv) (S) [D] (II) (ii) (Q)

Sol. **D**

Let $f(x) = x + \log_e x - x\log_e x$, $x \in (0, \infty)$.

- Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
- Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
- Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

41. Let $f : (0, \infty) \to R$ be given by $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$, then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$ (B) $f(x)$ is monotonically decreasing on (0, 1)

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$ (D) $f(2^x)$ is an odd function of x on R

50. If the function $e^{-x} f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true ?

(A) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$

(B) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(C) $f'(x) < f(x), 0 < x < \frac{1}{4}$

(D) $f'(x) < f(x), \frac{3}{4} < x < 1$

Sol. **C**

Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$.

49. Which of the following is true for $0 < x < 1$?

(A) $0 < f(x) < \infty$

(B) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(C) $-\frac{1}{4} < f(x) < 1$

(D) $-\infty < f(x) < 0$

Sol. **(D)**

If $f(x) = \dfrac{x^2}{2 - 2\cos x}$; $g(x) = \dfrac{x^2}{6x - 6\sin x}$ where $0 < x < 1$, then

(A) both 'f' and 'g' are increasing functions
(B) 'f' is decreasing & 'g' is increasing function
(C*) 'f' is increasing & 'g' is decreasing function
(D) both 'f' & 'g' are decreasing function

Let $a + b = 4$, where $a < 2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x, prove that

$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases. [JEE '97, 5]

Find all the values of the parameter 'a' for which the function ;
$f(x) = 8ax - a \sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$.

Find the set of values of 'a' for which the function,

$f(x) = \left(1 - \dfrac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \sqrt{7}$ is increasing at every point of its domain.

Find the values of 'a' for which the function $f(x) = \sin x - a \sin 2x - \frac{1}{3}\sin 3x + 2ax$ increases throughout the number line.

Find the set of all values of the parameter 'a' for which the function,
$f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$.

Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

12. Let f and g be increasing and decreasing functions, respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - h(x)$ is

(a) always zero
(b) always negative
(c) always positive
(d) strictly increasing

(a) $(-\infty, 0]$; **(b)** $\uparrow$ in $\left(1,\frac{5}{3}\right)$ and $\downarrow$ in $(-\infty,1)\cup\left(\frac{5}{3},\infty\right)-\{-3\}$; **(c)** $x=\frac{5}{3}$;

(d) removable discont. at $x=-3$ (missing point) and non removable discont. at $x=1$ (infinite type)

(e) -2

Construct the graph of the function $f(x)=-\left|\frac{x^2-9}{x+3}-x+\frac{2}{x-1}\right|$ and comment upon the following

(a) Range of the function,

(b) Intervals of monotonocity,

(c) Point(s) where f is continuous but not diffrentiable,

(d) Point(s) where f fails to be continuous and nature of discontinuity.

Passage I

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

(2010)

11. The real numbers s lies in the interval

(a) $\left(-\frac{1}{4},0\right)$ (b) $\left(-11,-\frac{3}{4}\right)$ (c) $\left(-\frac{3}{4},-\frac{1}{2}\right)$ (d) $\left(0,\frac{1}{4}\right)$

12. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(a) $\left(\frac{3}{4},3\right)$ (b) $\left(\frac{21}{64},\frac{11}{16}\right)$ (c) $(9, 10)$ (d) $\left(0,\frac{21}{64}\right)$

13. The function $f'(x)$ is

(a) increasing in $\left(-t,-\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4},t\right)$

(b) decreasing in $\left(-t,-\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4},t\right)$

(c) increasing in $(-t, t)$

(d) decreasing in $(-t, t)$

18. Let $f(x) = \cot x - \tan x - 2\tan 2x - 4\tan 4x - 8\cot 8x$, $x \neq \frac{n\pi}{8}$, $n \in I$ and $g(x) = x^3 + 6x - 1$. One more function $h(x)$ is defined as $h: R - \left\{\frac{n\pi}{8}, n \in I\right\} \to R$, $h(x) = f(x) + g(x)$ then identify the correct statement(s).

(a) $h''\left(\frac{\pi}{24}\right) = \frac{\pi}{4}$

(b) $h(x)$ is odd function

(c) $h(x)$ is increasing in the domain.

(d) If the equation $h(x) = \lambda$ has a solution in $(0, 3)$ then number of integral values of λ is 7.

Show that the angle between the tangent at any point 'A' of the curve $\ln(x^2+y^2) = C\tan^{-1}\frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

Find the condition that the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ & $\frac{x^2}{a'}+\frac{y^2}{b'}=1$ may cut orthogonally.

A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q. Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ & $(x^2/a^2) + (y^2/b^2) = 1$ may touch if $c = a + b$.

The tangent at a variable point P of the curve $y = x^2 - x^3$ meets it again at Q. Show that the locus of the middle point of PQ is $y = 1 - 9x + 28x^2 - 28x^3$.

The curve $y = ax^3 + bx^2 + cx + 5$, touches the x - axis at P (– 2, 0) & cuts the y-axis at a point Q where its gradient is 3. Find a, b, c.

The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find 'a'.

If the tangent at the point (x_1, y_1) to the curve $x^3 + y^3 = a^3$ $(a \neq 0)$ meets the curve again in (x_2, y_2) then show that $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$.

Prove that the segment of the tangent to the curve $y = \frac{a}{2}\, ln\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}$ contained between the y-axis & the point of tangency has a constant length.

A straight line is drawn through the origin and parallel to the tangent to a curve $\frac{x+\sqrt{a^2-y^2}}{a} = ln\left(\frac{a+\sqrt{a^2-y^2}}{y}\right)$ at an arbitary point M. Show that the locus of the point P of intersection of the straight line through the origin & the straight line parallel to the x-axis & passing through the point M is $x^2 + y^2 = a^2$.

Example 31 The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$

(a) on the left of $x = c$ (b) on the right of $x = c$

(c) at no point (d) at all points

Ans. (a)

Problem 8. Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$.

Example 8. Prove that sum of intercepts of the ngent at any point to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on e coordinate axes is constant.

Example 8. Let f (x) be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6-x)$ and $f'(0) = 0 = f'(2) = f'(5)$. Determine the minimum number of zeroes of $g(x) = (f''(x))^2 + f'(x) f'''(x)$ in the interval [0, 6].

Problem 28. If f'' be continuous on [a, b] and derivable on (a, b), then prove that

$$f(b) - f(a) - \left(\frac{b-a}{2}\right)\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d)$$

for some real number d between a and b.

If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0\ \forall$ $x \in (a, b)$, then :

(a) $f''(c) < 0$ for some $c \in (a, b)$ (b) $f''(c) > 0\ \forall\ c \in (a, b)$

(c) $f(c) = 0$ for some $c \in (a, b)$ (d) none of these

If f be a continuous function on [0, 1], differentiable in (0, 1) such that $f(1) = 0$, then their exists some $c \in (0, 1)$ such that :

(a) $cf'(c) - f(c) = 0$ (b) $f'(c) + cf(c) = 0$

(c) $f'(c) - cf(c) = 0$ (d) $cf'(c) + f(c) = 0$

If $P(x) = (2013)\, x^{2012} - (2012)\, x^{2011} - 16x + 8$, then $P(x) = 0$ for $x \in \left[0, 8^{\frac{1}{2011}}\right]$ has :

(a) exactly one real root (b) no real root

(c) atleast one and at most two real roots (d) atleast two real roots

If $f(x)$ is continuous and differentiable in $[-3, 9]$ and $f'(x) \in [-2, 8]\ \forall\ x \in (-3, 9)$. Let N be t number of divisors of the greatest possible value of $f(9) - f(-3)$, then find the sum of digits N

2. Let $f:[0,8]\to R$ be differentiable function such that $f(0)=0, f(4)=1, f(8)=1$, then w
the following hold(s) good ?

(a) There exist some $c_1 \in (0,8)$ where $f'(c_1)=\frac{1}{4}$

(b) There exist some $c \in (0,8)$ where $f'(c)=\frac{1}{12}$

(c) There exist $c_1, c_2 \in [0,8]$ where $8f'(c_1)f(c_2)=1$

(d) There exist some $\alpha, \beta \in (0,2)$ such that $\int_0^8 f(t)\,dt = 3(\alpha^2 f(\alpha^3)+\beta^2 f(\beta^3))$

If $f(x)$ is a twice differentiable function such that $f(a)=0, f(b)=2, f(c)=-1, f(d)=2$ and $f(e)=0$, where $a<b<c<d<e$, then the minimum number of zeros of $g(x)=(f'(x))^2+f''(x)f(x)$ in the interval $[a,e]$ is ...... **Answer** 6.

17. If $f:R\to R$ is a differentiable function such that $f'(x)>2f(x)$ for all $x\in R$, and $f(0)=1$ then **(2017 Adv.)**

(a) $f(x)>e^{2x}$ in $(0,\infty)$

(b) $f'(x)<e^{2x}$ in $(0,\infty)$

(c) $f(x)$ is increasing in $(0,\infty)$

(d) $f(x)$ is decreasing in $(0\ \infty)$

17. For a polynomial g(x) with real coefficients, let m_g denote the number of distinct real roots of g(x). Suppose S in the set of polynomials with real coefficients defined by

$$S=\{(x^2-1)^2(a_0+a_1x+a_2x^2+a_3x^3): a_0, a_1, a_2, a_3 \in R\}.$$

For a polynomial f, let f′ and f″ denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'}+m_{f''})$, where $f\in S$, is _____

Q.4 For every twice differentiable function $f: R\to[-2,2]$ with $(f(0))^2+(f'(0))^2=85$, which of the following statement(s) is (are) TRUE ?

(A) There exist r, s ∈ R, where r < s, such that f is one-one on the open interval (r, s)

(B) There exists $x_0 \in (-4,0)$ such that $|f'(x_0)|\le 1$

(C) $\lim_{x\to\infty} f(x)=1$

(D) There exist $\alpha\in(-4,4)$ such that $f(\alpha)+f''(\alpha)=0$ and $f'(\alpha)\ne 0$

Sol. **A, B, D**

Q.43 Let $f: R\to(0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1) ?

[A] $e^x-\int_0^x f(t)\sin t\,dt$

[B] $x^9-f(x)$

[C] $f(x)+\int_0^{\pi/2} f(t)\sin t\,dt$

[D] $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

Sol. **B, D**

55. Let $f, g : [-1, 2] \to \mathbb{R}$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

43. For every pair of continuous functions $f, g : [0, 1] \to R$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is(are)

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

Q.40 If $f : R \to R$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then

[A] $f'(1) \le 0$

[B] $0 < f'(1) \le \frac{1}{2}$

[C] $\frac{1}{2} < f'(1) \le 1$

[D] $f'(1) > 1$

Sol. **D**

f (x) is differentiable function and g (x) is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c) \cdot g''(c) < 0$.

Let $f(x) = \begin{cases} x^{\alpha} \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if α =

(A) -2 (B) -1 (C) 0 (D) $\frac{1}{2}$

Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

Using L.M.V.T. or otherwise prove that difference of square root of two consecutive natural numbers greater than N^2 is less than $\frac{1}{2N}$.

Let a, b, c be three real number such that $a < b < c$, f (x) is continuous in [a, c] and differentiable in (a, c). Also f ' (x) is strictly increasing in (a, c). Prove that

$(c - b) f(a) + (b - a) f(c) > (c - a) f(b)$

Let $a > 0$ and f be continuous in $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

f (x) and g (x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

Let f defined on [0, 1] be a twice differentiable function such that, $|f''(x)| \le 1$ for all $x \in [0, 1]$. If $f(0) = f(1)$, then show that, $|f'(x)| < 1$ for all $x \in [0, 1]$

Let f (x) and g (x) be differentiable functions such that $f'(x) g(x) \ne f(x) g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

Let f be continuous on [a, b] and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

Prove that if f is differentiable on [a, b] and if $f(a) = f(b) = 0$ then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

Prove that the expression

$$f(x+h) = f(x) + hf'(x+\theta h) \qquad \text{where } 0 < \theta < 1$$

is an equivalent form of the mean-value theorem.
Find the value of θ in terms of x and h when $f(x) = x^3$.

Suppose that $f(2) = 1$ and $f'(x) \le 5$ for all values of x.

Determine a lower bound for $f(-2)$.

Determine an upper bound for $f(5)$.

Let f be continuous on the interval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

Example 3. Let f(x) and g(x) be two differentiable functions and f(2) = 8, g(2) = 0, f(4) = 10 and g(4) = 8, then prove that $g'(x) = 4f'(x)$ for atleast one $x \in (2, 4)$

Example 1. Find whether the functions $f(x) = x^2 - 2x + 3$ and $g(x) = x^3 - 7x^2 + 20x - 5$ satisfy the conditions of the Cauchy's Mean Value Theorem in the interval [1, 4] and find the corresponding value of c.

Example 7. Let $f: [0, \infty) \to [0, \infty)$ be a continuous and differentiable function. Then show that

$$(f(4) - f(2))\frac{(f(4))^2 + (f(2))^2 + f(2)f(4)}{3} = 2f^2(c)f'(c),$$

where $c \in (2, 4)$

Example 10. Using Intermediate Value Theorem prove that there exists a number x such that

$$x^{200} + \frac{1}{1+\sin^2 x} = 200.$$

xample 4. Use Intermediate Value Theorem to ow that the equation $2x^3 + x^2 - x + 1 = 5$ has a solution he interval [1, 2].

Example 7. Let f be a continuous function defined from [0, 1] to [0, 1] with range [0, 1]. Show that is some 'c' in [0, 1] such that $f(c) = 1 - c$.

Given that $a > b > c > d$ then prove

at the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ will

ıve two real and distinct roots.

35. If the function f: $[0,4] \to R$ is differentiable then show tha

(i) For $a, b \in (0,4)$, $(f(4))^2-(f(0))^2 = 8f'(a)f(b)$

For the function

$$f(x) = x\cos\frac{1}{x},\ x \geq 1, \qquad \textit{(2009)}$$

(a) for at least one x in the interval $[1, \infty)$, $f(x+2)-f(x)<2$

(b) $\lim_{x\to\infty} f'(x) = 1$

(c) for all x in the interval $[1, \infty)$, $f(x+2)-f(x)>2$

(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

22. In [0,1] Lagranges Mean Value theorem is NOT applicable to *(2003S)*

(a) $f(x) = \begin{cases} \frac{1}{2}-x & x<\frac{1}{2} \\ \left(\frac{1}{2}-x\right)^2 & x \geq \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x|x|$

(d) $f(x) = |x|$

Then, the quadratic equation $ax^2+bx+c=0$ has

(a) no root in (0,2)

(b) atleast one root in (1,2)

(c) a double root in (0, 2)

(d) two imaginary roots

8. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)dx$$

6. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

(a) at least one root in (0, 1)

(b) one root in (2, 3) and the other in (–2, – 1)

(c) imaginary roots

(d) None of the above

5. Let a, b, c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

(a) $\gamma = \frac{\alpha + \beta}{2}$

(b) $\gamma = \alpha + \frac{\beta}{2}$

(c) $\gamma = \alpha$

(d) $\alpha < \gamma < \beta$

6. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has

(2000, 1M)

(a) both roots in (a, b)

(b) both roots in $(-\infty, a)$

(c) both roots in $(b, +\infty)$

(d) one root in $(-\infty, a)$ and the other in (b, ∞)