2. By definition the region *inside* the parabola $y = x^2$ is the set of points (a, b) such that $b \geq a^2$. We are interested in those circles all of whose points are in this region. A *bubble at a point* P on the graph of $y = x^2$ is the *largest* such circle that contains P. (You may assume the fact that there is a *unique* such circle at any given point on the parabola.)

(a) A bubble at some point on the parabola has radius 1. Find the center of this bubble.

(b) Find the radius of the smallest possible bubble at some point on the parabola. Justify.

The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let a and b be the maximum and minimum values of y/x over all the points (x, y) on the ellipse. Find the value of $a + b$. [8 + 7 = 15]

27 Let the function $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \max\left\{\frac{|x-y|}{x+y+1} : 0 \leq y \leq 1\right\} \text{ for } 0 \leq x \leq 1.$$

Then which of the following statements is correct?

(A) f is strictly increasing on $\left[0, \frac{1}{2}\right]$ and strictly decreasing on $\left[\frac{1}{2}, 1\right]$.
(B) f is strictly decreasing on $\left[0, \frac{1}{2}\right]$ and strictly increasing on $\left[\frac{1}{2}, 1\right]$.
(C) f is strictly increasing on $\left[0, \frac{\sqrt{3}-1}{2}\right]$ and strictly decreasing on $\left[\frac{\sqrt{3}-1}{2}, 1\right]$.
(D) f is strictly decreasing on $\left[0, \frac{\sqrt{3}-1}{2}\right]$ and strictly increasing on $\left[\frac{\sqrt{3}-1}{2}, 1\right]$.

Passage Based Questions

Let $f(x) = (1-x)^2 \sin^2 x + x^2$, $\forall x \in R$ and
$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\, dt \ \forall x \in (1, \infty)$.

15. Consider the statements

P: There exists some $x \in R$ such that,
$f(x) + 2x = 2(1 + x^2)$.

Q: There exists some $x \in R$ such that,
$2f(x) + 1 = 2x(1 + x)$.

Then,

(a) both P and Q are true (b) P is true and Q is false
(c) P is false and Q is true (d) both P and Q are false

16. Which of the following is true?

(a) g is increasing on $(1, \infty)$
(b) g is decreasing on $(1, \infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

Find the value of $x > 1$ for which the function

$$F(x) = \int_x^{x^2} \frac{1}{t} \ln\left(\frac{t-1}{32}\right) dt$$ is increasing and decreasing.

Q.13 Let f be a real valued function defined on the interval $(0, \infty)$ by

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$$

Then which of the following statements is (are) true?

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$.

Prove that $f(x) = \int_2^{e^x} \left(9\cos^2(2\ln t) - 25\cos(2\ln t) + 17\right) dt$ is always an increasing function of x, $\forall\, x \in R$

For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S in the set of polynomials with real coefficients defined by

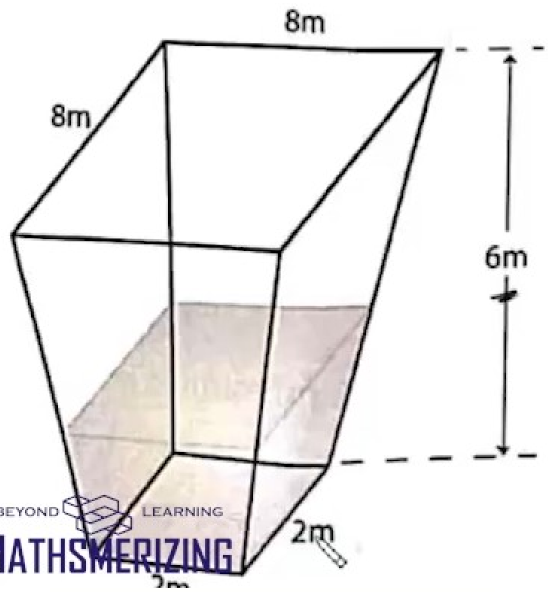
$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}.$$

For a polynomial f, let f′ and f″ denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

(b) Suppose the graph of $f(x)$ is being traced on a computer screen with the uniform speed of 1 cm per second (i.e., this is how fast the length of the curve is increasing). Show that at the moment the point corresponding to $x = 1$ is being drawn, the x coordinate is increasing at the rate of

$$\frac{1}{\sqrt{2 + \sin(2)}} \text{ cm per second.}$$

8. A pond has been dug at the Indian Statistical Institute as an inverted truncated pyramid with a square base (see figure below). The depth of the pond is 6m. The square at the bottom has side length 2m and the top square has side length 8m. Water is filled in at a rate of $\frac{19}{3}$ cubic meters per hour. At what rate is the water level rising exactly 1 hour after the water started to fill the pond?



10. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\frac{d^2 f(x)}{dx^2}$ is positive for all $x \in \mathbb{R}$, and suppose $f(0) = 1, f(1) = 4$. Which of the following is not a possible value of $f(2)$?

(A) 7. (B) 8. (C) 9. (D) 10.

Q3

$P(x) = x^n + a_1 x^{n-1} + ... + a_n$ is a polynomial with real coefficients. $a_1^2 < a_2$.
Prove that all roots of $P(x)$ cannot be real.

5. Consider the polynomial $p(x) = (x + a_1)(x + a_2) \cdots (x + a_{10})$ where a_i is a real number for each $i = 1, \ldots, 10$. Suppose all of the eleven coefficients of $p(x)$ are positive. For each of the following statements, decide if it is true or false. Write your answers as a sequence of four letters (T/F) in correct order.

(i) The polynomial $p(x)$ must have a global minimum. (ii) Each a_i must be positive.

(iii) All real roots of $p'(x)$ must be negative. (iv) All roots of $p'(x)$ must be real.

2. Let p, q and r be real numbers with $p^2 + q^2 + r^2 = 1$.

(a) Prove the inequality $3p^2q + 3p^2r + 2q^3 + 2r^3 \leq 2$.

(b) Also find the smallest possible value of $3p^2q + 3p^2r + 2q^3 + 2r^3$. Specify exactly when the smallest and the largest possible value is achieved.

30. Consider the function

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\right) e^{-x},$$

where $n \geq 4$ is a positive integer. Which of the following statements is correct?

(A) f has no local extremum

(B) For every n, f has a local maximum at $x = 0$

(C) f has no local extremum if n is odd and has a local maximum at $x = 0$ when n is even

(D) f has no local extremum if n is even and has a local maximum at $x = 0$ when n is odd.

7. Let a, b, c be three real numbers which are roots of a cubic polynomial, and satisfy $a + b + c = 6$ and $ab + bc + ac = 9$. Suppose $a < b < c$. Show that

$$0 < a < 1 < b < 3 < c < 4.$$

24. Let

$$p(x) = x^3 - 3x^2 + 2x,\ x \in \mathbb{R},$$

$$f_0(x) = \begin{cases} \int_0^x p(t)dt, & x \geq 0, \\ -\int_x^0 p(t)dt, & x < 0, \end{cases}$$

$$f_1(x) = e^{f_0(x)}, \quad f_2(x) = e^{f_1(x)}, \quad \ldots \quad , f_n(x) = e^{f_{n-1}(x)}.$$

How many roots does the equation $\frac{df_n(x)}{dx} = 0$ have in the interval $(-\infty, \infty)$?

(A) 1. (B) 3. (C) $n+3$. (D) $3n$.

2. For $x \geq 0$ define

$$f(x) = \frac{1}{x + 2\cos(x)}.$$

Determine the set $\{y \in \mathbb{R} : y = f(x), x \geq 0\}$.

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8. Let $t_1 < t_2 < \cdots < t_{99}$ be real numbers, and consider the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = | x - t_1 | + | x - t_2 | + \cdots + | x - t_{99} |$. Show that $\min_{x \in \mathbb{R}} f(x) = f(t_{50})$.

iv) f(x)=x^2.e^(-|x|)

$y = x^2 . e^{-|x|}$

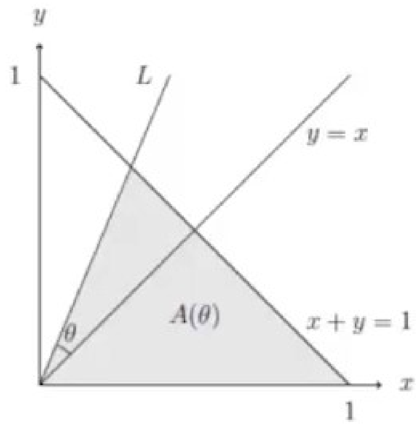
ii) f(x)= ex.ln(ex)

(i) A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ for all x. A function is said to be *odd* if $f(-x) = -f(x)$ for all x.

(a) What symmetry does the graph $y = f(x)$ of an even function have?
What symmetry does the graph $y = f(x)$ of an odd function have?

(b) Use these symmetries to show that the derivative of an even function is an odd function, and that the derivative of an odd function is an even function. [You should not use the chain rule.]

(ii) For $-45° < \theta < 45°$, the line L makes an angle θ with the line $y = x$ as drawn in the figure below. Let $A(\theta)$ denote the area of the triangle which is bounded by the x-axis, the line $x + y = 1$ and the line L.



(a) Let $0 < \theta < 45°$. Arguing geometrically, explain why

$$A(\theta) + A(-\theta) = \frac{1}{2}.$$

(b) For $0 < \theta < 45°$, determine a formula for $A(\theta)$.

(c) Sketch the graph of $A(\theta)$ against θ for $-45° < \theta < 45°$.

(d) In light of the identity in part (ii)(a), what symmetry does the graph of $A(\theta)$

$$f(x) = \int_{\cos x}^{\sin x} (1 - t + 2t^3)\,dt \text{ has in } [0, 2\pi]$$

(A) a maximum at $\frac{\pi}{4}$ & a minimum at $\frac{3\pi}{4}$

(B*) a maximum at $\frac{3\pi}{4}$ & a minimum at $\frac{7\pi}{4}$

(C) a maximum at $\frac{5\pi}{4}$ & a minimum at $\frac{7\pi}{4}$

(D) neither a maxima nor minima

20 Let $f : [0, 2] \to \mathbb{R}$ be a continuous function such that

$$\frac{1}{2}\int_0^2 f(x)dx < f(2).$$

Then which of the following statements must be true?

(A) f must be strictly increasing.

(B) f must attain a maximum value at $x = 2$.

(C) f cannot have a minimum at $x = 2$.

(D) None of the above.

$$f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases} \text{ and } g(x) = \int_0^x f(t)dt \text{ , } x \in [1, 3] \text{ then } g(x) \text{ has}$$

(A) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$
(B) local maxima at $x = 1$ and local minima at $x = 2$
(C) no local maxima
(D) no local minima

C is the complex number. $f: C \to R$ is defined by $f(z) = |z^3 - z + 2|$. What is the minimum value of f on the unit circle $|z| = 1$?

Find the minimum value of $(\sec^{-1} x)^2 + (\text{cosec}^{-1} x)^2$.

2. Let a be a fixed real number. Consider the equation

$$(x+2)^2(x+7)^2 + a = 0, \ x \in R,$$

where R is the set of real numbers. For what values of a, will the equation have exactly one double-root?

4. Let $\alpha > 0$. If the equation $p(x) = x^3 - 9x^2 + 26x - \alpha$ has three positive real roots, then α must satisfy

(A) $\alpha \le 27$.

(B) $\alpha > 81$.

(C) $27 < \alpha \le 54$.

(D) $54 < \alpha \le 81$.

16. Let $ABCD$ be a rectangle with its shorter side $a > 0$ units and perimeter $2s$ units. Let $PQRS$ be any rectangle such that vertices A, B, C and D respectively lie on the lines PQ, QR, RS and SP. Then the maximum area of such a rectangle $PQRS$ in square units is given by

(A) s^2 (B) $2a(s-a)$ (C) $\frac{s^2}{2}$ (D) $\frac{5}{2}a(s-a)$.

30. The number of distinct real roots of the equation $x\sin(x)+\cos(x) = x^2$ is

(A) 0 (B) 2 (C) 24 (D) none of the above.

viii: Functions of the form min(f(x), g(x)) or max(f(x), g(x))

$f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$

If $f(x) = x + \int_0^1 [xy^2 + x^2y] f(y)\, dy$ where x and y are independent variable. Find f(x).

A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \le h \le 1} \Delta(h)$,

then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$

6. (a) Compute $\frac{d}{dx}\left[\int_0^{e^x} \log(t)\cos^4(t)\, dt\right]$. [4 marks]

(b) For $x > 0$ define $F(x) = \int_1^x t\log(t)dt$. [6 marks]

i. Determine the open interval(s) (if any) where $F(x)$ is decreasing and the open interval(s) (if any) where $F(x)$ is increasing.

ii. Determine all the local minima of $F(x)$ (if any) and the local maxima of $F(x)$ (if any) .

(b) Let f be a differentiable function defined on a subset A of the real numbers. Define

$$f^*(y) := \max_{x \in A} \{yx - f(x)\},$$

whenever the above maximum is finite.

For the function $f(x) = -\ln(x)$, determine the set of points for which f^* is defined and find an expression for $f^*(y)$ involving only y and constants. [5 marks]

Find the area enclosed by

$\max\{|x|, |Y|\} < 2$ and $||x| - |y|| > 1$