Rolle's theorem

If a real-valued function *f* is

1. continuous on a proper closed interval $[a, b]$,
2. differentiable on the open interval (a, b),
3. and $f(a) = f(b)$,

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$

Or at least one root of the equation $f'(c) = 0$ will lie in (a, b)

Example 8. Let f (x) be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6-x)$ and $f'(0) = 0 = f'(2) = f'(5)$. Determine the minimum number of zeroes of $g(x) = (f''(x))^2 + f'(x) f'''(x)$ in the interval [0, 6].

Problem 28. If f'' be continuous on [a, b] and derivable on (a, b), then prove that

$$f(b) - f(a) - \left(\frac{b-a}{2}\right)\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d)$$

for some real number d between a and b.

If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0$ $\forall$ $x \in (a, b)$, then :

(a) $f''(c) < 0$ for some $c \in (a, b)$
(b) $f''(c) > 0$ $\forall$ $c \in (a, b)$
(c) $f(c) = 0$ for some $c \in (a, b)$
(d) none of these

If f be a continuous function on [0, 1], differentiable in (0, 1) such that $f(1) = 0$, then their exists some $c \in (0, 1)$ such that :

(a) $c f'(c) - f(c) = 0$
(b) $f'(c) + c f(c) = 0$
(c) $f'(c) - c f(c) = 0$
(d) $c f'(c) + f(c) = 0$

If $P(x) = (2013)\, x^{2012} - (2012)\, x^{2011} - 16x + 8$, then $P(x) = 0$ for $x \in \left[0, 8^{\frac{1}{2011}}\right]$ has :

(a) exactly one real root
(b) no real root
(c) atleast one and at most two real roots
(d) atleast two real roots

If $f(x)$ is continuous and differentiable in $[-3, 9]$ and $f'(x) \in [-2, 8]\ \forall\ x \in (-3, 9)$. Let N be t
number of divisors of the greatest possible value of $f(9) - f(-3)$, then find the sum of digits
N

2. Let $f:[0, 8] \to R$ be differentiable function such that $f(0) = 0, f(4) = 1, f(8) = 1$, then
the following hold(s) good ?

(a) There exist some $c_1 \in (0, 8)$ where $f'(c_1) = \dfrac{1}{4}$

(b) There exist some $c \in (0, 8)$ where $f'(c) = \dfrac{1}{12}$

(c) There exist $c_1, c_2 \in [0, 8]$ where $8f'(c_1)f(c_2) = 1$

(d) There exist some $\alpha, \beta \in (0, 2)$ such that $\int_0^8 f(t)\,dt = 3(\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3))$

If $f(x)$ is a twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2$ and $f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeros of $g(x) = (f'(x))^2 + f''(x)f(x)$ in the interval $[a, e]$ is ...... **Answer** 6.

17. If $f : R \to R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$ then **(2017 Adv.)**

(a) $f(x) > e^{2x}$ in $(0, \infty)$

(b) $f'(x) < e^{2x}$ in $(0, \infty)$

(c) $f(x)$ is increasing in $(0, \infty)$

(d) $f(x)$ is decreasing in $(0\ \infty)$

17. For a polynomial g(x) with real coefficients, let m_g denote the number of distinct real roots of g(x). Suppose S in the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}.$$

For a polynomial f, let f′ and f″ denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Q.4 For every twice differentiable function $f : R \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ?

(A) There exist r, s ∈ R, where r < s, such that f is one-one on the open interval (r, s)

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(C) $\lim_{x\to\infty} f(x) = 1$

(D) There exist $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

Sol. **A, B, D**

Q.43 Let $f : R \to (0, 1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1) ?

[A] $e^x - \int_0^x f(t)\sin t\,dt$

[B] $x^9 - f(x)$

[C] $f(x) + \int_0^{\pi/2} f(t)\sin t\,dt$

[D] $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

Sol. **B, D**

55. Let $f, g : [-1, 2] \to \mathbb{R}$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

43. For every pair of continuous functions $f, g : [0, 1] \to R$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is(are)

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

Q.40 If $f : R \to R$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then

[A] $f'(1) \le 0$ [B] $0 < f'(1) \le \frac{1}{2}$

[C] $\frac{1}{2} < f'(1) \le 1$ [D] $f'(1) > 1$

Sol. **D**

f (x) is differentiable function and g (x) is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c) \cdot g''(c) < 0$.

Let $f(x) = \begin{cases} x^{\alpha} \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$

(A) -2 (B) -1 (C) 0 (D) $\frac{1}{2}$

Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

Using L.M.V.T. or otherwise prove that difference of square root of two consecutive natural numbers greater than N^2 is less than $\frac{1}{2N}$.

Let a, b, c be three real number such that $a < b < c$, f (x) is continuous in [a, c] and differentiable in (a, c). Also $f'(x)$ is strictly increasing in (a, c). Prove that

$(c - b) f(a) + (b - a) f(c) > (c - a) f(b)$

Let $a > 0$ and f be continuous in $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

f (x) and g (x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

Let f defined on [0, 1] be a twice differentiable function such that, $|f''(x)| \le 1$ for all $x \in [0, 1]$ If $f(0) = f(1)$, then show that, $|f'(x)| < 1$ for all $x \in [0, 1]$

Let f (x) and g (x) be differentiable functions such that $f'(x) g(x) \ne f(x) g'(x)$ for any real x. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

Let f be continuous on [a, b] and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

Prove that if f is differentiable on [a, b] and if $f(a) = f(b) = 0$ then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

Lagrange's Mean Value Theorem (LMVT) or first mean value theorem.

If f(x) is a function, so that

1. f(x) is continuous on the close interval [a, b] and also
2. differentiable on the open interval (a, b),

then there is point c in (a, b) that is, $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Prove that the expression

$$f(x+h) = f(x) + hf'(x+\theta h) \qquad \text{where } 0 < \theta < 1$$

is an equivalent form of the mean-value theorem.
Find the value of θ in terms of x and h when $f(x) = x^3$.

Intermediate value theorem

The intermediate value theorem states that if *f* is a continuous function whose domain contains the interval [*a*, *b*], then it takes on any given value between *f*(*a*) and *f*(*b*) at some point within the interval.

Suppose that $f(2) = 1$ and $f'(x) \le 5$ for all values of x.

Determine a lower bound for $f(-2)$.

Determine an upper bound for $f(5)$.

Let f be continuous on the interval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

Example 3. Let f(x) and g(x) be two differentiable functions and f(2) = 8, g(2) = 0, f(4) = 10 and g(4) = 8, then prove that g'(x) = 4 f'(x) for atleast one $x \in (2, 4)$

Example 1. Find whether the functions $f(x) = x^2 - 2x + 3$ and $g(x) = x^3 - 7x^2 + 20x - 5$ satisfy the conditions of the Cauchy's Mean Value Theorem in the interval [1, 4] and find the corresponding value of c.

Example 7. Let $f:[0,\infty)\to[0,\infty)$ be a continuous and differentiable function. Then show that

$$(f(4)-f(2))\frac{(f(4)^2+(f(2))^2+f(2)f(4)}{3}=2f^2(c)f'(c),$$

where $c\in(2,4)$

Example 10. Using Intermediate Value Theorem prove that there exists a number x such that

$$x^{200}+\frac{1}{1+\sin^2 x}=200.$$

xample 4. Use Intermediate Value Theorem to w that the equation $2x^3+x^2-x+1=5$ has a solution he interval [1, 2].

Example 7. Let f be a continuous function defined from [0, 1] to [0, 1] with range [0, 1]. Show that is some 'c' in [0, 1] such that $f(c)=1-c$.

Given that $a>b>c>d$ then prove at the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ will ve two real and distinct roots.

35. If the function $f:[0,4]\to R$ is differentiable then show tha

(i) For $a, b\in(0,4)$, $(f(4))^2-(f(0))^2=8f'(a)f(b)$

For the function

$$f(x)=x\cos\frac{1}{x},\ x\geq 1, \quad \textit{(2009)}$$

(a) for at least one x in the interval $[1,\infty)$, $f(x+2)-f(x)<2$

(b) $\lim_{x\to\infty} f'(x)=1$

(c) for all x in the interval $[1,\infty)$, $f(x+2)-f(x)>2$

(d) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

22. In [0,1] Lagranges Mean Value theorem is NOT applicable to **(2003S)**

(a) $f(x)=\begin{cases}\frac{1}{2}-x & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2 & x\ge\frac{1}{2}\end{cases}$

(b) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\ne 0\\ 1, & x=0\end{cases}$

(c) $f(x)=x|x|$

(d) $f(x)=|x|$

Then, the quadratic equation $ax^2+bx+c=0$ has

(a) no root in (0,2)
(b) atleast one root in (1,2)
(c) a double root in (0, 2)
(d) two imaginary roots

8. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)dx$$

6. If $a+b+c=0$, then the quadratic equation $3ax^2+2bx+c=0$ has

(a) at least one root in (0, 1)
(b) one root in (2, 3) and the other in (–2, – 1)
(c) imaginary roots
(d) None of the above

5. Let a, b, c be real numbers, $a \ne 0$. If α is a root of $a^2x^2+bx+c=0$, β is the root of $a^2x^2-bx-c=0$ and $0<\alpha<\beta$, then the equation $a^2x^2+2bx+2c=0$ has a root γ that always satisfies

(a) $\gamma=\frac{\alpha+\beta}{2}$
(b) $\gamma=\alpha+\frac{\beta}{2}$
(c) $\gamma=\alpha$
(d) $\alpha<\gamma<\beta$

6. If $b>a$, then the equation $(x-a)(x-b)-1=0$ has **(2000, 1M)**

(a) both roots in (a, b)
(b) both roots in $(-\infty, a)$
(c) both roots in $(b, +\infty)$
(d) one root in $(-\infty, a)$ and the other in (b, ∞)

Cauchy’s Mean Value Theorem or extended mean value theorem

if the functions f and g are both continuous on the closed interval [a,b] and differentiable on the open interval (a, b), then there exists some c in the open interval (a, b) , such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Geometrical meaning of Cauchy's theorem

if $g(a) \neq g(b)$ and $g'(c) \neq 0$, this is equivalent to:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Bolzano's theorem

If a continuous function has values of opposite sign inside an interval, then it has at least one root in that interval.

Let f(x) be a continuous function defined in an interval [a,b].

Then, if $f(a)\cdot f(b) < 0$ (therefore, $f(a) < 0$ and $f(b) > 0$ or $f(a) > 0$ and $f(b) < 0$), there exists at least a point c inside the interval (a,b) such that $f(c)=0$.